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Notes on the diamond-α dynamic derivative on time scales

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Abstract

Various dynamic derivative formulae have been proposed in the development of a time scales calculus, with the goal of unifying continuous and discrete analysis. Recent discussions of combined dynamic derivatives, in particular the $\diamond_\alpha$ derivative defined as a linear combination of the Δ and the ∇ derivatives, have promised improved approximation formulae for computational applications. This paper presents an equivalent definition of the $\diamond_\alpha$ functions without reference to the existing Δ and ∇ derivatives, examines the status of the $\diamond_\alpha$ as a dynamic derivative and its properties relative to the Δ and ∇ derivatives, and compares data obtained using the various dynamic derivatives as approximation formulae in computational experiments. A $\diamond_\alpha$ integral case is investigated.

Keywords: Dynamic derivatives; Combined formula; Dynamic equations; Approximations; Nonuniform time scales

1. Introduction

Much of the development of time scales theory has focused on the unification of continuous and discrete analytical methods. Recent discussions have suggested that the theory and methods of time scales might also provide a means of integrating difference and differential methods for modeling nonlinear systems of dynamic equations on domains that are arbitrary nonempty closed subsets of the reals. To this end, the usefulness of various dynamic derivative formulae,

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including the standard Δ and ∇ derivatives, in approximating functions and solutions of nonlinear differential equations has been explored [3,4,6,10]. It has been demonstrated in several recent papers [8–10] that a proposed dynamic derivative formula, called the $\diamondsuit_\alpha$ derivative and defined as a linear combination, or the Broyden's formula [5,13], of the Δ and the ∇ dynamic derivatives, provides a more accurate approximation to the conventional derivative. The question remains, however, as to whether the $\diamondsuit_\alpha$ derivative is a well-defined dynamic derivative upon which a calculus on time scales can be built.

This paper redefines the $\diamondsuit_\alpha$ derivative independently of the standard Δ and ∇ dynamic derivatives, and further examines its properties and relationship with the Δ and the ∇ formulae. In addition, we examine the feasibility of formulating a corresponding $\diamondsuit_\alpha$ integral. Finally, we implement several computational experiments and compare the performance of various dynamic derivatives as approximation formulae.

Our discussions will be organized as follows: Section 2 contains basic definitions and theorems of time scales theory and of the Δ and ∇ dynamic derivatives. In Section 3, we define the $\diamondsuit_\alpha$ derivative without reference to the Δ and ∇ derivatives, and show that this new function is well-defined and equivalent to a linear combination of the Δ and ∇ derivatives at points where those derivatives exist. We present several theorems concerning the properties of the $\diamondsuit_\alpha$ derivative. In Section 4, we consider two counterexamples that demonstrate that a $\diamondsuit_\alpha$ antiderivative does not exist for some continuous functions on a time scale in the case of a fixed α value strictly between 0 and 1.

Finally, in Section 5, we discuss computational experiments where nonuniform time scales resulting from adaptive computations of the numerical solution of a solitary wave equation are employed [11,12]. Numerical errors will be computed and compared between different first order dynamic derivative approximates over an interval which includes a singularity in the conventional derivative. The simulation results confirm the computational superiority of the diamond-α as an approximation formula. The combined dynamic derivatives can be used in various nonlinear dynamic equations generated via adaptive or hybrid approximations [6,12].

2. The delta and nabla derivatives

An one-dimensional time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$ and has the inherited topology. Let $a = \inf\mathbb{T}$ and $b = \sup\mathbb{T}$. For $t \in \mathbb{T}$ such that $a < t < b$, we define the *forward-jump operator*, σ, and *backward-jump operator*, ρ, as

$$\sigma(t) = \inf\{s \in \mathbb{T}\colon\ s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T}\colon\ s < t\},$$

respectively, and

$$\sigma(b) = b, \qquad \rho(a) = a,$$

if $\mathbb{T}$ is bounded. The corresponding *forward-step* and *backward-step functions* μ, η are defined as

$$\mu(t) = \sigma(t) - t, \qquad \eta(t) = t - \rho(t),$$

respectively. For a function f defined on $\mathbb{T}$, to provide a shorthand notation we let

$$f^\sigma(t) = f\bigl(\sigma(t)\bigr), \qquad f^\rho(t) = f\bigl(\rho(t)\bigr).$$

We say that a point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$ and *left-scattered* if $\rho(t) < t$. A point $t \in \mathbb{T}$ that is both right-scattered and left-scattered is called *scattered*. Also, we say that a point $t \in \mathbb{T}$ is *right-dense* if $\sigma(t) = t$, *left-dense* if $\rho(t) = t$, and *dense* if it is both right-dense and left-dense.

We define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{b\}$ if $\mathbb{T}$ is bounded above and b is left-scattered; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Similarly, we define $\mathbb{T}_\kappa = \mathbb{T} \setminus \{a\}$ if $\mathbb{T}$ is bounded below and a is right-scattered; otherwise $\mathbb{T}_\kappa = \mathbb{T}$. We denote $\mathbb{T}^\kappa \cap \mathbb{T}_\kappa$ by $\mathbb{T}^\kappa_\kappa$. We say a time scale $\mathbb{T}$ is *uniform* if for all $t \in \mathbb{T}^\kappa_\kappa$, $\mu(t) = \eta(t)$. A uniform time scale is an interval if $\mu(t) = 0$, and is a uniform difference grid if $\mu(t) > 0$.

We say a function f defined on $\mathbb{T}$ is right continuous at $t \in \mathbb{T}$ if for all $\epsilon > 0$ there is some $\delta > 0$ such that for all $s \in [t, t+\delta)$, $|f(t) - f(s)| < \epsilon$. Similarly, we say that f is left continuous at $t \in \mathbb{T}$ if for all $\epsilon > 0$ there is some $\delta > 0$ such that for all $s \in (t-\delta, t]$, $|f(t) - f(s)| < \epsilon$. The function $f(t)$ is said to be continuous if it is both right and left continuous.

For the sake of readability of subsequent formulas, we introduce the following notation. Let $t, s \in \mathbb{T}$ and define

$$\mu_{ts} = \sigma(t) - s, \qquad \eta_{ts} = \rho(t) - s.$$

Let $f : \mathbb{T} \to \mathbb{R}$ be a function on a time scale. Then for $t \in \mathbb{T}^\kappa$ we define $f^\Delta(t)$ to be the value, if one exists, such that for all $\epsilon > 0$ there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that for all $s \in U$

$$\bigl|\bigl[f^\sigma(t) - f(s)\bigr] - f^\Delta(t)\bigl(\sigma(t) - s\bigr)\bigr| < \epsilon\bigl|\sigma(t) - s\bigr|.$$

We say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. Similarly, for $t \in \mathbb{T}_\kappa$ we define $f^\nabla(t)$ to be the number, if one exists, such that for all $\epsilon > 0$ there is a neighborhood V of t such that for all $s \in V$

$$\bigl|\bigl[f^\rho(t) - f(s)\bigr] - f^\nabla(t)\bigl(\rho(t) - s\bigr)\bigr| < \epsilon\bigl|\rho(t) - s\bigr|.$$

We say that f is *nabla differentiable* on $\mathbb{T}_\kappa$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

In subsequence proofs, we will wish to make use of the following theorem due to Hilger [7], and the analogous theorem for the nabla case which can be found in [2,3]:

Theorem 2.1. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following*:

(i) *If f is delta differentiable at t, then f is continuous at t.*
(ii) *If f is left continuous at t and t is right-scattered, then f is delta differentiable at t with*

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}.$$

(iii) *If t is right-dense, then f is delta differentiable at t iff the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Theorem 2.2. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_\kappa$. Then we have the following*:

(i) *If f is nabla differentiable at t, then f is continuous at t.*
(ii) *If f is right continuous at t and t is left-scattered, then f is nabla differentiable at t with*

$$f^\nabla(t) = \frac{f(t) - f^\rho(t)}{t - \rho(t)}.$$

(iii) *If t is left-dense, then f is nabla differentiable at t iff the limit*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. In this case

$$f^{\nabla}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

With the above theorems in hand we can establish the following corollary.

Corollary 2.3. *Assume $f:\mathbb{T}\to\mathbb{R}$ is a function and let $t\in\mathbb{T}^{\kappa}_{\kappa}$. The existence of the delta derivative of f at t does not imply the existence of the nabla derivative at t, and vice versa.*

Proof. Consider the function

$$f(t)=\begin{cases} t\sin(1/t), & t\neq 0,\\ 0, & t=0,\end{cases}$$

on a time scale $\mathbb{T}=[-2,-1]\cup[0,1]$. The function f is continuous at 0, and the point $0\in\mathbb{T}$ is right-dense, left-scattered. By 2.2(ii), f is nabla differentiable at 0. But a finite limit

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

does not exist at 0. Thus by 2.1(iii), f is not delta differentiable at 0. To show the existence of the delta derivative does not imply the existence nabla derivative, we may consider the same function f at point 0 on a time scale $\mathbb{T}=[-1,0]\cup[1,2]$. □

3. The diamond-α dynamic derivative

Definition 3.1. Let $\mathbb{T}$ be a time scale. We define $f^{\diamond_\alpha}(t)$ to be the value, if one exists, such that for all $\epsilon>0$ there is a neighborhood U of t (i.e., $U=(t-\delta,t+\delta)\cap\mathbb{T}$ for some $\delta>0$) such that for all $s\in U$

$$\left|\alpha\left[f^{\sigma}(t)-f(s)\right]\eta_{ts}+(1-\alpha)\left[f^{\rho}(t)-f(s)\right]\mu_{ts}-f^{\diamond_\alpha}(t)\mu_{ts}\eta_{ts}\right|<\epsilon|\mu_{ts}\eta_{ts}|.$$

We say that f is *diamond-α differentiable* on $\mathbb{T}^{\kappa}_{\kappa}$ provided $f^{\diamond_\alpha}(t)$ exists for all $t\in\mathbb{T}^{\kappa}$.

Remark. It is clear that $f^{\diamond_\alpha}(t)$ reduces to $f^{\Delta}(t)$ for $\alpha=1$ and $f^{\nabla}(t)$ for $\alpha=0$. The idea of such a formula can be traced back to Broyden's method in which combinations of different formulae are utilized. The new formula takes advantage of each individual method and provides a far more effective formula [5].

We show that the function described above is well-defined. Let each of $\Phi_1(t)$ and $\Phi_2(t)$ be values such that $\forall\epsilon>0$ there exist neighborhoods U_1 and U_2 of t such that $\forall s\in U_1$

$$\left|\alpha\left[f^{\sigma}(t)-f(s)\right]\eta_{ts}+(1-\alpha)\left[f^{\rho}(t)-f(s)\right]\mu_{ts}-\Phi_1(t)\mu_{ts}\eta_{ts}\right|<\epsilon|\mu_{ts}\eta_{ts}|$$

and $\forall s\in U_2$

$$\left|\alpha\left[f^{\sigma}(t)-f(s)\right]\eta_{ts}+(1-\alpha)\left[f^{\rho}(t)-f(s)\right]\mu_{ts}-\Phi_2(t)\mu_{ts}\eta_{ts}\right|<\epsilon|\mu_{ts}\eta_{ts}|.$$

Let $\epsilon>0$ be given and set $\epsilon_*=\epsilon/2$. Then $\forall s\in U=U_1\cap U_2$

$$\begin{aligned}
&\big|\Phi_1(t)-\Phi_2(t)\big||\mu_{ts}\eta_{ts}| \\
&\quad =\big|\Phi_1(t)\mu_{ts}\eta_{ts}-\Phi_2(t)\mu_{ts}\eta_{ts}\big| \\
&\quad =\big|-\alpha\big[f^\sigma(t)-f(s)\big]\eta_{ts}-(1-\alpha)\big[f^\rho(t)-f(s)\big]\mu_{ts}+\Phi_1(t)\mu_{ts}\eta_{ts} \\
&\qquad +\alpha\big[f^\sigma(t)-f(s)\big]\eta_{ts}+(1-\alpha)\big[f^\rho(t)-f(s)\big]\mu_{ts}-\Phi_2(t)\mu_{ts}\eta_{ts}\big| \\
&\quad <\big|\alpha\big[f^\sigma(t)-f(s)\big]\eta_{ts}+(1-\alpha)\big[f^\rho(t)-f(s)\big]\mu_{ts}-\Phi_1(t)\mu_{ts}\eta_{ts}\big| \\
&\qquad +\big|\alpha\big[f^\sigma(t)-f(s)\big]\eta_{ts}+(1-\alpha)\big[f^\rho(t)-f(s)\big]\mu_{ts}-\Phi_2(t)\mu_{ts}\eta_{ts}\big| \\
&\quad <\epsilon_*|\mu_{ts}\eta_{ts}|+\epsilon_*|\mu_{ts}\eta_{ts}| \\
&\quad =\epsilon|\mu_{ts}\eta_{ts}|.
\end{aligned}$$

Thus $|\Phi_1(t)-\Phi_2(t)|<\epsilon$ and letting ϵ go to zero, we see $\Phi_1(t)=\Phi_2(t)$.

Theorem 3.2. *Let $0\leqslant\alpha\leqslant 1$. If f is both Δ and ∇ differentiable at $t\in\mathbb{T}$, then f is $\diamondsuit_\alpha$ differentiable at t and $f^{\diamondsuit_\alpha}(t)=\alpha f^\Delta(t)+(1-\alpha)f^\nabla(t)$.*

Proof. Assume $f^\Delta(t)$ and $f^\nabla(t)$ exist. Then $\forall\epsilon>0$, $\exists$ neighborhoods U_1 and U_2 such that $\forall s\in U_1$

$$\big|\big[f^\sigma(t)-f(s)\big]-f^\Delta(t)\mu_{ts}\big|<\epsilon|\mu_{ts}|$$

and $\forall s\in U_2$

$$\big|\big[f^\rho(t)-f(s)\big]-f^\nabla(t)\eta_{ts}\big|<\epsilon|\eta_{ts}|.$$

Then $\forall s\in U_1$

$$\big|\alpha\big[f^\sigma(t)-f(s)\big]\eta_{ts}-\alpha f^\Delta(t)\mu_{ts}\eta_{ts}\big|<\alpha\epsilon|\mu_{ts}\eta_{ts}|$$

and $\forall s\in U_2$

$$\big|(1-\alpha)\big[f^\rho(t)-f(s)\big]\mu_{ts}-(1-\alpha)f^\nabla(t)\mu_{ts}\eta_{ts}\big|<(1-\alpha)\epsilon|\mu_{ts}\eta_{ts}|.$$

Thus $\forall s\in U=U_1\cap U_2$ we have

$$\begin{aligned}
&\big|\alpha\big[f^\sigma(t)-f(s)\big]\eta_{ts}+(1-\alpha)\big[f^\rho(t)-f(s)\big]\mu_{ts}-\big[\alpha f^\Delta(t)+(1-\alpha)f^\nabla(t)\big]\mu_{ts}\eta_{ts}\big| \\
&\quad \leqslant\big|\alpha\big[f^\sigma(t)-f(s)\big]\eta_{ts}-\alpha f^\Delta(t)\mu_{ts}\eta_{ts}\big| \\
&\qquad +\big|(1-\alpha)\big[f^\rho(t)-f(s)\big]\mu_{ts}-(1-\alpha)f^\nabla(t)\mu_{ts}\eta_{ts}\big| \\
&\quad <\alpha\epsilon|\mu_{ts}\eta_{ts}|+(1-\alpha)\epsilon|\mu_{ts}\eta_{ts}|=\epsilon|\mu_{ts}\eta_{ts}|.
\end{aligned}$$

Thus $f^{\diamondsuit_\alpha}(t)$ exists and $f^{\diamondsuit_\alpha}(t)=\alpha f^\Delta(t)+(1-\alpha)f^\nabla(t)$. □

Corollary 3.3. *Let $t\in\mathbb{T}$ be dense. Then if $f'(t)$ exists we have*

$$f^{\diamondsuit_\alpha}(t)=f^\Delta(t)=f^\nabla(t)=f'(t).$$

Proof. Let the point t be dense and $f'(t)=\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$ exist as a finite value. For a sufficiently small neighborhood U of t, $\forall s,t\in U$ we may substitute $h=s-t$ to see $f'(t)=\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$. Then by Theorem 2.1(iii), $f^\Delta(t)=f'(t)$, and by Theorem 2.2(iii), $f^\nabla(t)=f'(t)$. Thus by Theorem 3.2,

$$f^{\diamondsuit_\alpha}(t)=\alpha f^\Delta(t)+(1-\alpha)f^\nabla(t)=\alpha f'(t)+(1-\alpha)f'(t)=f'(t). \qquad \square$$

Lemma 3.4. *Let $t \in \mathbb{T}$ be scattered. Then f is continuous at t.*

Proof. Assume $t \in \mathbb{T}$ is scattered. Then $\mu(t) > 0$ and $\eta(t) > 0$. Let $0 < \delta < \min(\mu(t), \eta(t))$. Then $\forall \epsilon > 0$ there is a neighborhood $U = (t-\delta, t+\delta) \cap \mathbb{T}$ of t such that $\forall s \in U$, $s = t$ and thus $|f(t) - f(s)| = 0 < \epsilon$. $\square$

Corollary 3.5. *Let $t \in \mathbb{T}$ be scattered. Then*

(i) *$f^{\Delta}(t)$ exists and*
$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t};$$
(ii) *$f^{\nabla}(t)$ exists and*
$$f^{\nabla}(t) = \frac{f^{\rho}(t) - f(t)}{\rho(t) - t};$$
(iii) *$f^{\diamond_\alpha}(t)$ exists and*
$$f^{\diamond_\alpha}(t) = \alpha \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t} + (1-\alpha)\frac{f^{\rho}(t) - f(t)}{\rho(t) - t}.$$

Proof. By Lemma 3.4, f is continuous at t. Then (ii) follows from Theorem 2.1(ii), and (iii) follows from Theorem 2.2(ii). Then by Theorem 3.2,
$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t) = \alpha \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t} + (1-\alpha)\frac{f^{\rho}(t) - f(t)}{\rho(t) - t}. \qquad \square$$

Corollary 3.6. *Let $t \in \mathbb{T} \subset \mathbb{R}$ be left-scattered, right-dense, and assume*
$$f'(t^+) = \lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h}$$
exists. Then

(i) $f^{\Delta}(t) = f'(t^+)$;
(ii) $f^{\nabla}(t) = \frac{f^{\rho}(t) - f(t)}{\rho(t) - t}$;
(iii) $f^{\diamond_\alpha}(t) = \alpha f'(t^+) + (1-\alpha)\frac{f^{\rho}(t) - f(t)}{\rho(t) - t}$.

Proof. For all neighborhoods $U = (t-\delta, t+\delta)$ of t such that $\delta < t - \rho(t)$, we have $\forall s, t \in U$, $s - t > 0$. Thus we can substitute $h = s - t$ in the limit from the right to see
$$\lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h} = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$
Then by Theorem 2.1(iii),
$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t^+).$$
Since $f'(t^+)$ exists, (ii) follows from Theorem 2.2(ii). Then by Theorem 3.2,
$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t) = \alpha f'(t^+) + (1-\alpha)\frac{f^{\rho}(t) - f(t)}{\rho(t) - t}. \qquad \square$$

The proof of the following corollary is similar.

Corollary 3.7. *Let $t \in \mathbb{T} \subset \mathbb{R}$ be left-dense, right-scattered, and assume*

$$f'(t^-) = \lim_{h \to 0^-} \frac{f(t+h) - f(t)}{h}$$

exists. Then

(i) $f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t}$;
(ii) $f^{\nabla}(t) = f'(t^-)$;
(iii) $f^{\diamond_\alpha}(t) = \alpha \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t} + (1-\alpha) f'(t^-)$.

Theorem 3.8. *Let $\mathbb{T}$ be a time scale and $0 \leqslant \alpha \leqslant 1$. If f is $\diamond_\alpha$ differentiable at t, then f is continuous at t.*

Proof. Assume f is $\diamond_\alpha$ differentiable at $t \in \mathbb{T}$. If t is a dense or scattered point, the result follows from Corollaries 3.3 and 3.5, respectively. It remains to consider the two cases where t is right-dense and left-scattered, or t is right-scattered and left-dense.

Assume t right-dense and left-scattered. Thus $\sigma(t) = t$ and $\rho(t) < t$.

Let $\epsilon \in (0, 1)$ and

$$\epsilon_* = \frac{\epsilon\alpha|\rho(t) - t|}{(1-\alpha)|[f^{\rho}(t) - f(t)] - f^{\diamond_\alpha}(t)[\alpha(\rho(t) - t) - 1]| + |\rho(t) - t| + 1}.$$

Thus $0 < \epsilon_* < 1$. Then there is a neighborhood U_1 of t such that for all $s \in U_1$

$$\begin{aligned}
&\bigl|\alpha\bigl[f^{\sigma}(t) - f(s)\bigr]\eta_{ts} + (1-\alpha)\bigl[f^{\rho}(t) - f(s)\bigr]\mu_{ts} - f^{\diamond_\alpha}(t)\mu_{ts}\eta_{ts}\bigr| \\
&\quad = \bigl|\alpha\bigl[f(t) - f(s)\bigr]\bigl[\bigl(\rho(t) - t\bigr) + (t-s)\bigr] \\
&\qquad + (1-\alpha)\bigl[\bigl(f^{\rho}(t) - f(t)\bigr) + \bigl(f(t) - f(s)\bigr)\bigr](t-s) \\
&\qquad - f^{\diamond_\alpha}(t)(t-s)\bigl[\bigl(\rho(t) - t\bigr) + (t-s)\bigr]\bigr| \\
&\quad = \bigl|\alpha\bigl[f(t) - f(s)\bigr]\bigl(\rho(t) - t\bigr) + (1-\alpha)\bigl[f^{\rho}(t) - f(t)\bigr](t-s) \\
&\qquad + \bigl[f(t) - f(s)\bigr](t-s) - f^{\diamond_\alpha}(t)(t-s)\bigl[\bigl(\rho(t) - t\bigr) + (t-s)\bigr]\bigr| \\
&\quad = \bigl|\bigl[f(t) - f(s)\bigr]\bigl[\alpha\bigl(\rho(t) - t\bigr) + (t-s)\bigr] \\
&\qquad + \bigl[(1-\alpha)\bigl[f^{\rho}(t) - f(t)\bigr] - f^{\diamond_\alpha}(t)\bigl[\alpha\bigl(\rho(t) - t\bigr) + (t-s)\bigr]\bigr](t-s)\bigr| \\
&\quad < \epsilon_*|\mu_{ts}\eta_{ts}| \\
&\quad = \epsilon_*\bigl|(t-s)\bigr|\bigl|\bigl[\bigl(\rho(t) - t\bigr) + (t-s)\bigr]\bigr|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\bigl|\bigl|\bigl[f(t) - f(s)\bigr]\bigl[\alpha\bigl(\rho(t) - t\bigr) + (t-s)\bigr]\bigr| \\
&\qquad - \bigl|\bigl[(1-\alpha)\bigl[f^{\rho}(t) - f(t)\bigr] - f^{\diamond_\alpha}(t)\bigl[\alpha\bigl(\rho(t) - t\bigr) + (t-s)\bigr]\bigr](t-s)\bigr|\bigr| \\
&\quad < \epsilon_*\bigl|(t-s)\bigr|\bigl|\bigl[\bigl(\rho(t) - t\bigr) + (t-s)\bigr]\bigr|.
\end{aligned}$$

Since t left-scattered, right-dense we have for all $s \in U_1$, $\rho(t) < t \leqslant s$. Thus for all $s \in U = U_1 \cap (t - \epsilon_*, t + \epsilon_*)$

$$\begin{aligned}
&\big|\big[f(t)-f(s)\big]\alpha\big(\rho(t)-t\big)\big| \\
&\quad< \big|\big[f(t)-f(s)\big]\big[\alpha\big(\rho(t)-t\big)+(t-s)\big]\big| \\
&\quad< \big|(1-\alpha)\big[f^{\rho}(t)-f(t)\big]-f^{\diamondsuit_\alpha}(t)\big[\alpha\big(\rho(t)-t\big)+(t-s)\big]\big||t-s| \\
&\qquad+\epsilon_*|t-s|\big|\big(\rho(t)-t\big)+(t-s)\big| \\
&\quad< \epsilon_*\big|(1-\alpha)\big[f^{\rho}(t)-f(t)\big]-f^{\diamondsuit_\alpha}(t)\big[\alpha\big(\rho(t)-t\big)-1\big]\big|+\epsilon_*\big[\big|\rho(t)-t\big|+1\big].
\end{aligned}$$

Thus

$$\begin{aligned}
\big|f(t)-f(s)\big| &< \frac{\epsilon_*[|(1-\alpha)[f^{\rho}(t)-f(t)]-f^{\diamondsuit_\alpha}(t)[\alpha(\rho(t)-t)-1]|+|\rho(t)-t|+1]}{\alpha|\rho(t)-t|} \\
&=\epsilon. \qquad \square
\end{aligned}$$

Theorem 3.9. *Let $\mathbb{T}$ be a time scale and $0<\alpha<1$. If f is $\diamondsuit_\alpha$ differentiable at t, then f is both Δ and ∇ differentiable at t.*

Proof. Let $\mathbb{T}$ be a time scale and $0<\alpha<1$. Let $\epsilon>0$ be given, and set $\epsilon_*=\epsilon\frac{1-\alpha}{1+\alpha}>0$. Assume f is $\diamondsuit_\alpha$ differentiable at $t\in\mathbb{T}$. Thus by Theorem 3.8, f is continuous at t. If t is a dense or scattered point, the result follows from Corollaries 3.3 and 3.5, respectively. It remains to consider the two cases where t is right-dense and left-scattered, or t is right-scattered and left-dense.

Assume t right-scattered and left-dense. Thus $\sigma(t)>t$ and $\rho(t)=t$. Also, since f is continuous at t, by Theorem 2.1(ii), f is Δ differentiable at t. Then for all $\epsilon_*>0$ there is a neighborhood U_1 of t such that for all $s\in U_1$

$$\big|\alpha\big[f^{\sigma}(t)-f(s)\big]\eta_{ts}+(1-\alpha)\big[f^{\rho}(t)-f(s)\big]\mu_{ts}-f^{\diamondsuit_\alpha}(t)\mu_{ts}\eta_{ts}\big|<\epsilon_*|\mu_{ts}\eta_{ts}|$$

and neighborhood U_2 of t such that for all $s\in U_2$

$$\big|\big[f^{\sigma}(t)-f(s)\big]-f^{\Delta}(t)\mu_{ts}\big|<\epsilon_*|\mu_{ts}|.$$

Choose γ such that $f^{\diamondsuit_\alpha}(t)=\alpha f^{\Delta}(t)+(1-\alpha)\gamma$. Then there exists neighborhood $U=U_1\cap U_2$ of t such that for all $s\in U$

$$\begin{aligned}
&\big|\alpha\big[f^{\sigma}(t)-f(s)\big]\eta_{ts}+(1-\alpha)\big[f^{\rho}(t)-f(s)\big]\mu_{ts}-\big[\alpha f^{\Delta}(t)+(1-\alpha)\gamma\big]\mu_{ts}\eta_{ts}\big| \\
&\quad=\big|\alpha\big[f^{\sigma}(t)-f(s)-f^{\Delta}(t)\mu_{ts}\big]\eta_{ts}+(1-\alpha)\big[f^{\rho}(t)-f(s)-\gamma\eta_{ts}\big]\mu_{ts}\big| \\
&\quad<\epsilon_*|\mu_{ts}\eta_{ts}|.
\end{aligned}$$

Thus

$$\begin{aligned}
\big|(1-\alpha)\big[f^{\rho}(t)-f(s)-\gamma\eta_{ts}\big]\mu_{ts}\big| &\leqslant \epsilon|\mu_{ts}\eta_{ts}|+\big|\alpha\big[f^{\sigma}(t)-f(s)-f^{\Delta}(t)\mu_{ts}\big]\eta_{ts}\big| \\
&<\epsilon_*|\mu_{ts}\eta_{ts}|+\alpha\epsilon_*|\mu_{ts}\eta_{ts}|=(1+\alpha)\epsilon_*|\mu_{ts}\eta_{ts}|.
\end{aligned}$$

Then

$$\big|\big[f^{\rho}(t)-f(s)\big]-\gamma\eta_{ts}\big|<\epsilon_*\frac{1+\alpha}{1-\alpha}|\eta_{ts}|=\epsilon|\eta_{ts}|.$$

Thus $f^{\nabla}(t)=\gamma$ exists.

The case t right-dense, left-scattered is similar. $\square$

Remark. Note that the strict inequalities in $0<\alpha<1$ are necessary for the results above. In the case $\alpha=1$, the $\diamondsuit_\alpha$ derivative reduces to the Δ derivative, which by Corollary 2.3, does not imply the existence of the ∇. Similarly for $\alpha=0$.

4. A diamond-α integral

We present two problematic cases that arise when we attempt to determine a corresponding $\diamondsuit_\alpha$ integral.

First, let $\alpha = \frac{1}{2}$ and $\mathbb{T}$ be the set $\{0, 1, 2, 3\}$. Then the $\diamondsuit_\alpha$ derivative for a function on $\mathbb{T}$ is defined on the set $\mathbb{T}^\kappa_\kappa$ which is $\{1, 2\}$. Define the function $f(t) \equiv 0$. Next define functions F and G as follows:

$$\begin{aligned} &F(0) = 0, \qquad G(0) = 1;\\ &F(1) = 5, \qquad G(1) = -3;\\ &F(2) = 0, \qquad G(2) = 1;\\ &F(3) = 5, \qquad G(3) = -3. \end{aligned}$$

Then

$$F^{\diamondsuit_\alpha}(1) = \frac{1}{2}\frac{F(2) - F(1)}{2-1} + \frac{1}{2}\frac{F(1) - F(0)}{1-0} = \frac{1}{2}(0-5) + \frac{1}{2}(5-0) = 0 = f(1)$$

and

$$F^{\diamondsuit_\alpha}(2) = \frac{1}{2}\frac{F(3) - F(2)}{3-2} + \frac{1}{2}\frac{F(2) - F(1)}{2-1} = \frac{1}{2}(5-0) + \frac{1}{2}(0-5) = 0 = f(2).$$

Also

$$G^{\diamondsuit_\alpha}(1) = \frac{1}{2}\frac{G(2) - G(1)}{2-1} + \frac{1}{2}\frac{G(1) - G(0)}{1-0} = \frac{1}{2}\big(1-(-3)\big) + \frac{1}{2}(-3-1) = 0 = f(1)$$

and

$$G^{\diamondsuit_\alpha}(2) = \frac{1}{2}\frac{G(3) - G(2)}{3-2} + \frac{1}{2}\frac{G(2) - G(1)}{2-1} = \frac{1}{2}(-3-1) + \frac{1}{2}\big(1-(-3)\big) = 0 = f(2).$$

Thus $F^{\diamondsuit_\alpha}(t) = G^{\diamondsuit_\alpha}(t) = f(t)$ on $\mathbb{T}^\kappa_\kappa$. We see that both F and G are $\diamondsuit_\alpha$ antiderivatives of f on $\mathbb{T}^\kappa_\kappa$. However,

$$\int_1^2 f(t)\diamondsuit_\alpha t = F(2) - F(1) = -5 \neq 4 = G(2) - G(1) = \int_1^2 f(t)\diamondsuit_\alpha t$$

and we have arrived at a contradiction.

The above counterexample can be generalized for any fixed α strictly between 0 and 1, and for any purely discrete time scale, such as $\mathbb{T} = \mathbb{Z}$.

Next, we present an example where no $\diamondsuit_\alpha$ antiderivative exists. Again let $\alpha = 1/2$. Let $\mathbb{T}$ be $(-\infty, 1] \cup [2, \infty)$. Set

$$f(t) = \begin{cases} -1, & x \leqslant 1, \\ 5, & x \geqslant 2. \end{cases}$$

Assume a $\diamondsuit_\alpha$ antiderivative F of f exists on $\mathbb{T}^\kappa_\kappa$. On $(-\infty, 1]$, F must be of the form $-t + C_1$ where C_1 is a constant. On $[2, \infty)$, F must be of the form $5t + C_2$. It follows therefore

$$F^{\diamondsuit_\alpha}(1) = \frac{1}{2}F^{\Delta}(1) + \frac{1}{2}F^{\nabla}(1) = f(1).$$

Thus

$$\frac{1}{2}\big[\big(5(2) + C_2\big) - \big(-1(1) + C_1\big)\big] + \frac{1}{2}(-1) = -1. \tag{4.1}$$

Also,

$$F^{\diamond_\alpha}(2) = \frac{1}{2}F^{\Delta}(2) + \frac{1}{2}F^{\nabla}(2) = f(2).$$

Thus

$$\frac{1}{2}(5) + \frac{1}{2}\big[\big(5(2) + C_2\big) - \big(-1(1) + C_1\big)\big] = 5. \tag{4.2}$$

From (4.1) and (4.2) we obtain a system of equations

$$C_1 - C_2 = 12,$$
$$C_1 - C_2 = 6$$

with no solution. Thus for function f, which is continuous on $\mathbb{T}$, no $\diamond_\alpha$ antiderivative exists on $\mathbb{T}_\kappa^\kappa$.

5. Numerical examples

We consider adaptive approximations of the kink function [12],

$$u(x, y) = \alpha \arctan \exp\big\{\beta - \sqrt{x^2 + y^2}\,\big\}, \tag{5.1}$$

for initializing a sequence of circular ring solitons from the sine-Gordon equation,

$$w_{rr} = w_{xx} + w_{yy} - \phi(x, y)\sin w, \quad r > 0,$$

in the spacial domain $\Omega = \{(x, y) \mid -a < x < a,\ -b < y < b\}$, where the function ϕ is often interpreted as a Josephson current density of the solitary wave [1,12].

For the sake of simplicity in our one-dimensional experiments, we set $\alpha = \beta = \pi$, $a = 14$ and $y \equiv 0$. Replace the notation x by t, from (5.1) we have

$$u(t) = \pi \arctan \exp\big\{\pi - |t|\big\}, \quad -14 \leqslant t \leqslant 14.$$

The function value changes rapidly while $-7 < t < 7$, and u is not smooth throughout the interval $[-14, 14]$ due to the fact that

$$u'(t) = \begin{cases} \pi/[1 + (\pi + t)^2], & t < 0, \\ -\pi/[1 + (\pi - t)^2], & t > 0, \end{cases}$$

and $u'(0)$ does not exist. The change of the derivative function value is more violent throughout the domain, and introduces substantial difficulties in approximating u' over the interval $[-14, 14]$ using one formula. This motivates our numerical investigations targeted at the approximations of u'.

In Fig. 1, we show the solitary kink function, its sine mode representation $s = \sin(u/2)$, the velocity of the kink, that is, $v = u'$, and an arc-length adaptive step (equivalent to μ or η) distribution generated based on the derivative function [11,12]. Note that the sudden decrease of the step sizes as t approaches 0 is due to the singularity involved. The adaptive mechanism established offers a nonuniform time scale $\mathbb{T}$ superimposed over the interval $[-14, 14]$ for a possibly more accurate approximation to the derivative function u' from the data u. The number of grids used, n, is 280, with the minimal step size $h_{\min} \approx 0.03473491$ and maximal step size $h_{\max} \approx 0.10497856$. Since the nonuniform grids obtained are symmetric about $t = 0$, anti-symmetric properties of the

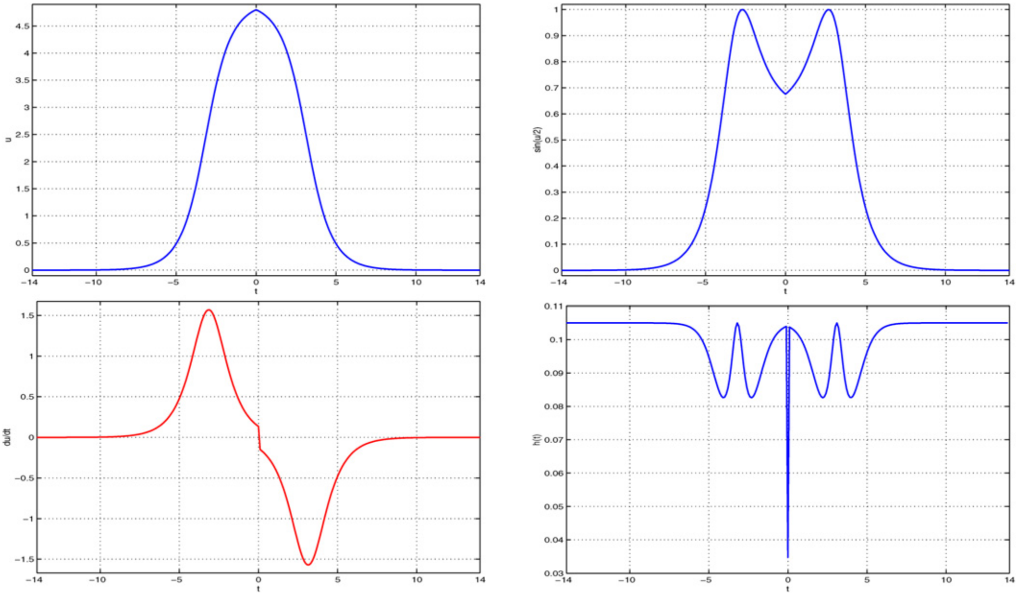

Fig. 1. The kink function u (top-left); its sine mode representation $\sin(u/2)$ (top-right); derivative u' (bottom-left); and distribution of the adaptive step sizes based on the derivative function (bottom-right).

Δ and ∇ dynamic derivatives are expected [6,8,9]. However, the phenomenon will not affect the overall accuracy of the approximation formulae.

Numerical errors of approximations of u' on $\mathbb{T}$ via different dynamic derivative

$$\epsilon_\Delta = u^\Delta - u',$$
$$\epsilon_\nabla = u^\nabla - u',$$
$$\epsilon_{\diamond_\alpha} = u^{\diamond_\alpha} - u'$$

are presented in Fig. 2. A modified finite difference formula,

$$u^{\mathrm{D}} = 2\frac{u^\Delta(t) - u^\nabla(t)}{\mu(t) + \eta(t)}, \tag{5.2}$$

is introduced for comparison purposes on the nonuniform discrete time scale $\mathbb{T}$. For it, we denote $u^{\mathrm{D}} - u' = \epsilon_{\mathrm{D}}$, $t \in \mathbb{T}$. Further, to see more precisely the superior quality of the $\diamond_{1/2}$ approximation, we also plot the pointwise relative errors,

$$\mathcal{E}_{\diamond_{1/2}} = \left\{\frac{|\epsilon_{\diamond_{1/2}}|_i}{|u'|_i}\right\}_{i=1}^n, \qquad \mathcal{E}_{\mathrm{D}} = \left\{\frac{|\epsilon_{\mathrm{D}}|_i}{|u'|_i}\right\}_{i=1}^n,$$

in Fig. 2. Logarithmic y-scale is used to give a better view of the details. All computations are implemented based on the u on the nonuniform discrete time scale $\mathbb{T}$.

It is observed in Fig. 2 that the $\diamond_\alpha$ dynamic derivative provides better overall approximation results than traditional Δ and ∇ dynamic derivatives with the α values used. When $\alpha = 1/2$, the $\diamond_\alpha$ derivative not only indicates a comparable quality as compared with the modified finite difference formula which is used in most adaptive algorithms, but also demonstrates a superior tolerance around the singular point. The latter property implies that the $\diamond_\alpha$ dynamic derivative

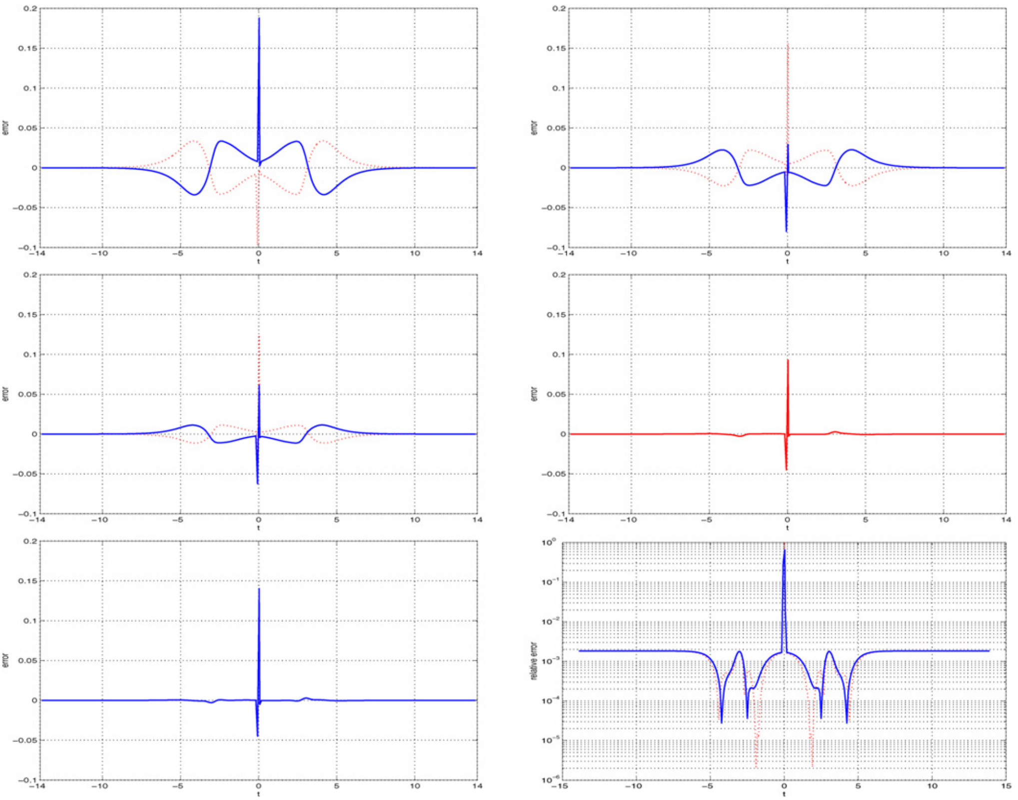

Fig. 2. Numerical errors of the different approximations of u' on the discrete time scale $\mathbb{T}$. Top-left: ϵ_{Δ} (dotted curve) and ϵ_{∇} (solid curve); top-right: $\epsilon_{\diamond_{1/6}}$ (dotted curve) and $\epsilon_{\diamond_{5/6}}$ (solid curve); center-left: $\epsilon_{\diamond_{1/3}}$ (dotted curve) and $\epsilon_{\diamond_{2/3}}$ (solid curve); center-right: $\epsilon_{\diamond_{1/2}}$; bottom-left: ϵ_{D}; bottom-right: relative errors of the $\diamond_{1/2}$ (solid curve) and modified central difference formula (5.2) (dotted curve). Logarithmic y-scale is used to show details of the error distributions.

is perhaps a better approximation formula to be used in numerical problems involving possible singularities. This is important in many adaptive and hybrid computational applications. We refer the reader to Table 1 for more detailed error data and interesting features displays based on enlarged $\epsilon_{\diamond_{1/2}}$ and ϵ_{D} values.

For each point t, it is possible to calculate a value of α that minimizes $|\epsilon_{\diamond_\alpha}|$. When t is scattered, we have

$$\epsilon_{\diamond_\alpha} = u^{\diamond_\alpha}(t) - u'(t) = u^{\diamond_\alpha}(t) = \alpha\frac{u^\sigma(t) - u(t)}{\sigma(t) - t} + (1-\alpha)\frac{u^\rho(t) - u(t)}{\rho(t) - t} - u'(t).$$

We find that the error is minimized when

$$\alpha = \frac{\mu(t)[\eta(t)u'(t) - u(t) + u^\rho(t)]}{\eta(t)[u^\sigma - u(t)] - \mu(t)[u(t) - u^\rho(t)]}.$$

Figure 3 presents the best α values for the grid points used in the domain of our kink function example. All our numerical experiments are carried out using MATLAB and SIMULINK subroutines on dual-processor DELL PRECISION workstations.

Table 1
A direct comparison of the numerical errors of $\epsilon_{\diamond_{1/2}}$ and ϵ_{D} when approximating the velocity of the kink function u. The values listed are 10^3 times the true errors

t	$\epsilon_{\diamond_{1/2}}$	ϵ_{D}	t	$\epsilon_{\diamond_{1/2}}$	ϵ_{D}
−10.850644	0.00259083	0.00259082	0.034601	92.74945926	140.25953222
−8.751093	0.02114102	0.02113858	0.069336	−2.63857717	−5.19001481
−6.652650	0.16921765	0.16814764	0.276496	−0.29481600	−0.28049321
−4.606311	0.42633868	0.20916658	0.482355	−0.34343580	−0.31842075
−3.295351	−1.91605181	−1.66914581	1.466492	−0.37749656	−0.16102783
−2.322998	0.23964928	0.60772959	2.322998	−0.23964928	−0.60772959
−1.466492	0.37749656	0.16102783	3.295351	1.91605182	1.66914581
−0.482355	0.34343580	0.31842075	4.606311	−0.42633868	−0.20916658
−0.276496	0.29481600	0.28049321	6.652650	−0.16921765	−0.16814764
−0.069336	−44.87149578	−44.92800153	8.751093	−0.02114102	−0.02113858

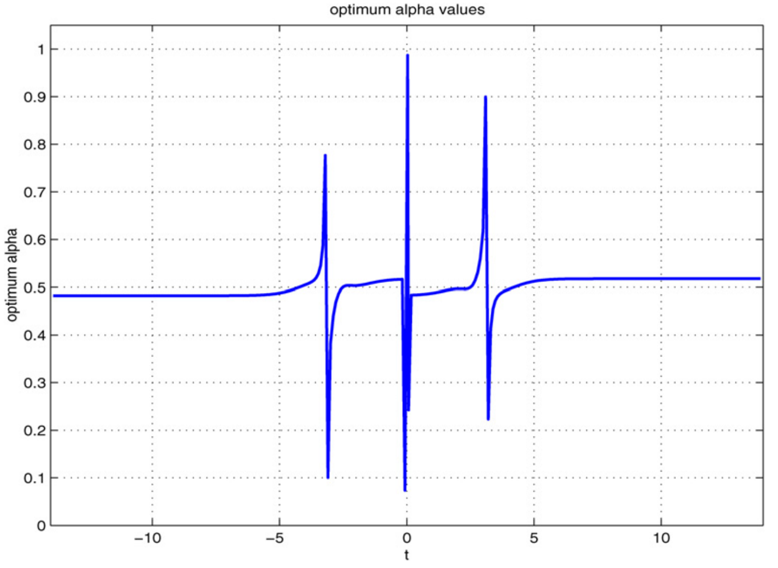

Fig. 3. Alpha values minimizing $|\epsilon_{\diamond_\alpha}|$. The optimized value of alpha is not only a function of the function considered, but also a function of the time scale used.

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