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Journal of Computational and Applied Mathematics 186 (2006) 391–415
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Partial dynamic equations on time scales

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Received 28 January 2004; received in revised form 24 January 2005

Abstract

In this work, we generalize existing ideas of the univariate case of the time scales calculus to the bivariate case. Formal definitions of partial derivatives and iterated integrals are offered, and bivariate partial differential operators are examined. In particular, solutions of the homogeneous and nonhomogeneous heat and wave operators are found when initial distributions given are in terms of elementary functions by means of the generalized Laplace Transform for the time scale setting. Finally, the so-termed mixed time scale setting is discussed. Examples are given and solutions are provided in tabular form.

Keywords: Time scale; Partial dynamic equation; Laplace transform; Multivariable calculus; Partial derivative; Iterated integral

1. Introduction and motivation

Throughout this work we assume a working knowledge of time scales calculus and the notation of time scales calculus. In particular, we assume knowledge of the univariate case of dynamic equations. For a treatment of the univariate case, see [2] and [3] or the Appendix (Section 5). At present, most of the work done in the time scales calculus has been in the univariate case. The notions of derivative and integral of a function of one variable are well established, and much of the theory of the continuous case has been generalized for arbitrary time scales. The ordinary dynamic equation (ODE) has been studied in depth and such concepts as boundary value problems (BVPs), initial value problems (IVPs), and differential operators in general are the current focus in the papers being written and published in

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2005.02.011

the area. To date two papers, namely Hoffacker [5] and Ahlbrandt and Morian [1], are known that have been published demonstrating the related ideas to the multivariate case and the study of partial dynamic equations (PDEs). Even in the discrete case, there is only one text known to be devoted to the study of partial difference equations, with the number of papers being written on the subject paling in comparison to its continuous counterpart. To us, this seems appauling because of the potential applications that are being overlooked. Indeed, in the only known text [4] on the subject, Cheng shows that discrete PDEs have much to offer in the way of applications. For example, he shows that the well known binomial coefficients are actually solutions of a PDE. He also argues that it is the discrete PDE that governs most population models that involve portions of the population migrating from one region to another during time periods. The applications to game theory are also astounding, especially to the mathematical biologist when he considers the applications of game theory to the mathematical theories of evolution in species.

We believe that one of the most important applications of this work will be to numerical analysis. Solutions to PDEs on arbitrary time scales in numerical terms amounts to an easy way of studying and finding solutions of the discretized equations from the continuous case. In particular, we believe that the most prominent advantages of studying time scales is that they will offer an effective alternative to the current way of finding solutions on nonuniformly spaced grids by adaptive methods, which often can be computationally intensive and slower to converge to required levels of accuracy.

2. Multivariable calculus on time scales

This section is devoted to the extension of the existing ideas of the time scales calculus to the multivariate case. Note that this is partly done in Ahlbrandt and Morian [1], and so similar ideas to many of the ones presented here can be found there. Thus, it is necessary to start with basic definitions. Consider the product $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, where $\mathbb{T}_i$ is a time scale for all $1 \leqslant i \leqslant n$. Then for any $\mathbf{t} \in \mathbb{T}$, with $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ for $t_i \in \mathbb{T}_i$ for all $1 \leqslant i \leqslant n$, define the following:

(i) the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by
$\sigma(\mathbf{t}) = (\sigma(t_1), \sigma(t_2), \ldots, \sigma(t_n))$, where $\sigma(t_i)$ represents the forward jump operator of $t_i \in \mathbb{T}_i$ on the time scale $\mathbb{T}_i$ for all $1 \leqslant i \leqslant n$. Hereafter, the forward jump operator of the time scale $\mathbb{T}_i$ for $t_i \in \mathbb{T}_i$ will be denoted by $\sigma(t_i) := \sigma_i(t)$.

(ii) the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by
$\rho(\mathbf{t}) = (\rho(t_1), \rho(t_2), \ldots, \rho(t_n))$, where $\rho(t_i)$ represents the backward jump operator of $t_i \in \mathbb{T}_i$ on the time scale $\mathbb{T}_i$ for all $1 \leqslant i \leqslant n$. Hereafter, the backward jump operator of the time scale $\mathbb{T}_i$ for $t_i \in \mathbb{T}_i$ will be denoted by $\rho(t_i) := \rho_i(t)$.

(iii) the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^n$ by
$\mu(\mathbf{t}) = (\mu(t_1), \mu(t_2), \ldots, \mu(t_n))$, where $\mu(t_i)$ represents the graininess function of $t_i \in \mathbb{T}_i$ on the time scale $\mathbb{T}_i$ for all $1 \leqslant i \leqslant n$. Again, from this point on the graininess function of the time scale $\mathbb{T}_i$ for $t_i \in \mathbb{T}_i$ will be denoted by $\mu(t_i) := \mu_i(t)$.

(iv) $\mathbb{T}^\kappa = \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa \times \cdots \times \mathbb{T}_n^\kappa$.

Having defined the multivariate time scale forward jump operator, the definition can be used to define the *partial Δ derivative* of a function $f(\mathbf{t})$. Before doing this, more notation is presented. From here on, set $f^{\sigma_i}(\mathbf{t}) = f(t_1, t_2, \ldots, t_{i-1}, \sigma_i(t), t_{i+1}, \ldots, t_n)$, and set

$f_i^s(\mathbf{t}) = f(t_1, t_2, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_n)$ (i.e. to evaluate $f_i^s(\mathbf{t})$, replace t_i in $f(\mathbf{t})$ by s).

Definition 1. Let $f:\mathbb{T}\to\mathbb{R}$ be a function and let $\mathbf{t}=(t_1,t_2,\ldots,t_i,\ldots,t_n)\in\mathbb{T}^{\kappa}$. Then define $f^{\Delta_i}(\mathbf{t})$ to be the number (provided it exists) with the property that given any $\varepsilon>0$, there exists a neighborhood U of t_i, with $U=(t_i-\delta,t_i+\delta)\cap\mathbb{T}_i$ for $\delta>0$ such that

$$|[f^{\sigma_i}(\mathbf{t})-f_i^s(\mathbf{t})]-f^{\Delta_i}(\mathbf{t})[\sigma_i(t)-s]|\leqslant\varepsilon|\sigma_i(t)-s| \text{ for all } s\in U.$$

f^{Δ_i} is called the *partial delta derivative* of *f* at *t* with respect to the variable t_i.

It is worth noting that this definition of the partial derivative states that to find the partial derivative with respect to t_i, treat the other variables as constants with respect to t_i, and take the usual delta derivative of $f(\mathbf{t})$ in the t_i variable on the time scale $\mathbb{T}_i$. Thus, the definition is just the generalization of its continuous analog, which follows from the fact that if $\mathbb{T}_i=\mathbb{R}$ for all *i*, then the partial delta derivative is the usual continuous partial derivative. Likewise, if $\mathbb{T}_i=h\mathbb{Z}$ for all *i*, then the partial delta derivative is the usual partial difference operator as given in Cheng [4]. With these observations, it is easy then to see that $f^{\Delta_{ij}}(\mathbf{t})$ (if this value exists) is found by first taking the partial derivative with respect to t_i to obtain $f^{\Delta_i}(\mathbf{t})$, and then taking the partial derivative of this derivative function with respect to t_j obtaining $f^{\Delta_{ij}}(\mathbf{t})$, so that $f^{\Delta_{ij}}=(f^{\Delta_i})^{\Delta_j}$. Higher order *mixed partials* are defined and evaluated similarly. The other notion that will be used is taking the partial derivative of the function $f(\mathbf{t})$ with respect to t_i *n* times (i.e. to evaluate $f^{\Delta_{ii\ldots i}}(\mathbf{t})$ where *i* occurs *n* times). From the discussion about mixed partials above, it follows that evaluating this derivative is equivalent to evaluating $f^{\Delta_i^n}(\mathbf{t})$, where Δ_i^n denotes taking the delta derivative with respect to t_i on the time scale $\mathbb{T}_i$ *n* times. Mixed partials often occur in a variety of orders. For example, given a function *f* of two variables t_1 and t_2, we may wish to take partials with respect to $t_1$first, then t_2, and then with respect to t_1. The notation for this would then be $f^{\Delta_{121}}$. We may wish to take partials with respect to t_1 twice and then once with respect to t_2. The notation for this situation is not clear from the discussion so far. It would most likely be thought that the notation would be $f^{\Delta_{12}^2}$, but this notation is confusing and unclear because it does not clearly indicate which partial(s) we are taking twice. Therefore, we will not adopt the simultaneous use of subscripts and superscripts in one derivative symbol when partials with respect to multiple variables are needed. Instead, a separate derivative symbol for each partial will be used. Thus, for example, to denote taking the partial of *f* with respect to t_1 twice and then taking the partial with respect to t_2, we write $f^{\Delta_1^2\Delta_2}$. In this spirit, if we wish to take partials of *f* with respect to t_1 twice, then with respect to t_2, and finally with respect to t_1 again, we would write $f^{\Delta_1^2\Delta_2\Delta_1}$. Finally, we will need to make use of the *order* of the partials being taken of a function. Define the order of the partial derivative to be the total number of partials with respect to all variables that are taken of the function. Thus, for example the order of $f^{\Delta_{121}}$ is three since three partials are taken, while $f^{\Delta_1^2\Delta_2^4}$ is of order six since six partials are taken. Before proceeding with the following example, we adopt a new convention. From this point on, unless otherwise noted, all appropriate regressivity conditions on the elementary functions will be assumed, and subscripts of elementary functions will be assumed to be constant with respect to the appropriate variables.

Example 1.

(1) Let $\mathbb{T}=\mathbb{R}\times h\mathbb{Z}\times\overline{q^{\mathbb{Z}}}$ and set $f(t_1,t_2,t_3)=t_1^4t_2^2t_3^3$. Then by Example 7, it follows that

$$f^{\Delta_{123}}(\mathbf{t})=4t_1^3(2t_2+h)(t_3^2(q^2+q+1))=8(q^2+q+1)t_1^3t_2t_3^2+4h(q^2+q+1)t_1^3t_3^2.$$

(2) Let $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3 \times \mathbb{T}_4$, where $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$, and $\mathbb{T}_4$ are arbitrary time scales. Set $f(\mathbf{t}) = t_1 e_{t_2}(t_3, 0)\sin_p(t_4, 0)$. We wish to compute $f^{\Delta_{1234}}(\mathbf{t})$, $f^{\Delta_4^3}(\mathbf{t})$, and $f^{\Delta_2^2}(\mathbf{t})$. To compute $f^{\Delta_{1234}}(\mathbf{t})$ and $f^{\Delta_2^2}(\mathbf{t})$, the derivative of $e_z(t,s)$ with respect to z must be known. Bohner and Peterson [2] show that this derivative is $(\int_s^t \frac{1}{1+\mu(\tau)z}\Delta\tau)e_z(t,s)$ for those $z \in \mathbb{C}$ satisfying $1+\mu(\tau)z \neq 0$ for τ between s and t. Thus,

$$
\begin{aligned}
f^{\Delta_{1234}}(\mathbf{t}) &= \left(e_{t_2}(t_3,0)\sin_p(t_4,0)\right)^{\Delta_{234}} \\
&= \left(\int_0^{t_3} \frac{1}{1+\mu_3(\tau)t_2}\,\Delta\tau e_{t_2}(t_3,0)\sin_p(t_4,0)\right)^{\Delta_{34}} \\
&= \left(\sin_p(t_4,0)\left(\frac{1}{1+\mu_3(t)t_2}\, e_{t_2}^{\sigma_3}(t,0) + t_2\int_0^{t_3}\frac{1}{1+\mu_3(\tau)t_2}\,\Delta\tau e_{t_2}(t_3,0)\right)\right)^{\Delta_4} \\
&= p\cos_p(t_4,0)\left(\frac{1}{1+\mu_3(t)t_2}\, e_{t_2}^{\sigma_3}(t,0) + t_2\int_0^{t_3}\frac{1}{1+\mu_3(\tau)t_2}\,\Delta\tau e_{t_2}(t_3,0)\right)
\end{aligned}
$$

$$
f^{\Delta_4^3}(\mathbf{t}) = -p^3 t_1 e_{t_2}(t_3,0)\sin_p(t_4,0)
$$

$$
\begin{aligned}
f^{\Delta_2^2}(\mathbf{t}) &= (f^{\Delta_2}(\mathbf{t}))^{\Delta_2} \\
&= \left(\left(\int_0^{t_3}\frac{1}{1+\mu_3(\tau)t_2}\,\Delta\tau\right) t_1 e_{t_2}(t_3,0)\sin_p(t_4,0)\right)^{\Delta_2} \\
&= \left(\int_0^{t_3}\frac{1}{1+\mu_3(\tau)t_2}\,\Delta\tau\right)^{\Delta_2} t_1 e_{\sigma_2(t)}(t_3,0)\sin_p(t_4,0) \\
&\quad + \left(\int_0^{t_3}\frac{1}{1+\mu_3(\tau)t_2}\Delta\tau\right)^2 t_1 e_{t_2}(t_3,0)\sin_p(t_4,0) \\
&= t_1\sin_p(t_4,0)\left[e_{\sigma_2(t)}(t_3,0)\int_0^{t_3}\left(\frac{1}{1+\mu_3(\tau)t_2}\right)^{\Delta_2}\Delta\tau \right.\\
&\quad \left. + e_{t_2}(t_3,0)\left(\int_0^{t_3}\frac{1}{1+\mu_3(\tau)t_2}\,\Delta\tau\right)^2\right] \\
&= t_1\sin_p(t_4,0)\left[e_{\sigma_2(t)}(t_3,0)\int_0^{t_3}\frac{-\mu_3(\tau)}{(1+\mu_3(\tau)t_2)(1+\mu_3(\tau)\sigma_2(t))}\Delta\tau \right.\\
&\quad \left. + e_{t_2}(t_3,0)\left(\int_0^{t_3}\frac{1}{1+\mu_3(\tau)t_2}\,\Delta\tau\right)^2\right]
\end{aligned}
$$

Note that in the computations above, the product rule, the quotient rule, the Fundamental Theorem of Calculus, and Lebesgue's Dominated Convergence Theorem were used in certain steps. Note that the Dominated Convergence is applicable in this case as the integral is a univariate integral.

The preceding discussion and the example itself serve to show that although partials on time scales are similar to the continuous case, in general the concept is much more complicated for arbitrary time scales.

The next point of concern for the multivariate case is the continuity of functions. Let $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, where $\mathbb{T}_i$ is a time scale for all $1 \leqslant i \leqslant n$. Recall that a time scale is given the topology that it inherits as a subset of $\mathbb{R}$ in the standard topology. Thus, giving the space $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$ the product topology or the subspace topology $\mathbb{T}$ inherits as a subspace of $\mathbb{R}^n$ is equivalent, as is well known from point-set topology.

Next, the bivariate *iterated integral* is presented.

Definition 2. Denote all partitions of the interval $[a_1, b_1]$ by $\mathscr{P}_1(a_1, b_1)$. $\mathscr{P}_{1_\delta}(a_1, b_1)$ is the set of all $P_1 \in \mathscr{P}_1(a_1, b_1)$ such that for every $\delta > 0$ and for each $i \in \{1, 2, \ldots, n\}$ either

$$t_i - t_{i-1} \leqslant \delta$$

or

$$t_i - t_{i-1} > \delta \text{ and } \rho(t_i) = t_{i-1}.$$

(This of course means that P_1 is given by $a = t_0 < t_1 < \cdots < t_n = b$.) $\mathscr{P}_2(a_2, b_2)$ and $\mathscr{P}_{2_\delta}(a_2, b_2)$ are defined similarly.

Definition 3. Let f be a bounded function on the rectangular region $[a_1, b_1] \times [a_2, b_2]$ as a subset of $\mathbb{T}_1 \times \mathbb{T}_2$, and let $P_1 \in \mathscr{P}_1(a_1, b_1)$, $P_2 \in \mathscr{P}_2(a_2, b_2)$ be given by $a_1 = t_0 < t_1 < \cdots < t_n = b_1$ and $a_2 = x_0 < x_1 < \cdots < x_m = b_2$, respectively. In each interval $[t_{i-1}, t_i)$ and $[x_{j-1}, x_j)$ with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, choose arbitrary points ξ_i and η_j and form the double sum

$$S = \sum_{j=1}^{m} \sum_{i=1}^{n} f(\xi_i, \eta_j)(t_i - t_{i-1})(x_j - x_{j-1}).$$

Then, just as in the univariate case, S is a *Riemann Δ-sum* of f corresponding to the partitions $P_1 \in \mathscr{P}_1(a_1, b_1)$ and $P_2 \in \mathscr{P}_2(a_2, b_2)$. f is *Riemann integrable* on $[a_1, b_1] \times [a_2, b_2]$ if there exists a number I with the property that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|S - I| < \varepsilon$$

for every Riemann Δ-sum S of f corresponding to any $P_1 \in \mathscr{P}_{1_\delta}(a_1, b_1)$ and $P_2 \in \mathscr{P}_{2_\delta}(a_2, b_2)$ and independent of the choice of $\xi_i \in [t_{i-1}, t_i)$ and $\eta_j \in [x_{j-1}, x_j)$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. The number I is called the *iterated Riemann Δ integral* of f on the region $[a_1, b_1] \times [a_2, b_2]$ and is denoted by

$$I = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(t_1, t_2) \Delta_1 \Delta_2,$$

where Δ_1 denotes integration with respect to t_1 and Δ_2 denotes integration with respect to t_2.

Note that with the definition above, in effect what we are doing is the same idea as for the partial derivative: hold one variable constant and integrate with respect to the second variable. This is why the double integral given above is called the iterated integral, as two univariate integrals are evaluated to obtain the double integral. The one thing that may not be clear in the definition is the choice of δ for the two partitions. This can be clarified by understanding the double integral as two iterated single integrals, since for the double integral to exist, both univariate integrals must exist. Thus, there exists a δ_1 corresponding to the integral with respect to t_1 that satisfies the First Cauchy Criterion for integrability in the univariate case, and likewise, there exists a δ_2 corresponding to the integral with respect to t_2 satisfying the same criterion in the univariate case. Thus, if both univariate integrals exist, then the double integral exists by choosing $\delta = \min\{\delta_1, \delta_2\}$. Next, note that to change the order of integration, one simply changes the order of the summation. Thus, to compute

$$I = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(t_1, t_2) \Delta_2 \Delta_1$$

we just compute the double sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_i, \eta_j)(t_i - t_{i-1})(x_j - x_{j-1}).$$

It also worth noting that the definition above can easily be extended to the n-dimensional case in a very obvious way. Iterated univariate integrals allow for the theory from the univariate case, such as the Fundamental Theorem of Calculus, to extend quite easily. We now state two of these theorems without proofs as their proofs follow by holding one variable constant and then integrating with respect to the other variable.

Theorem 1. *Every bivariate continuous function f on* $[a_1, b_1] \times [a_2, b_2]$ *is* Δ*-integrable.*

Theorem 2 (*Bivariate Fundamental Theorem of Calculus*). *Let* $g(t_1, t_2)$ *be continuous on* $[a_1, b_1] \times [a_2, b_2]$ *and the single and mixed* Δ*-partials exist on* $[a_1, b_1) \times [a_2, b_2)$. *If* $g^{\Delta_{12}}$ *is* Δ*-integrable on* $[a_1, b_1] \times [a_2, b_2]$, *then*

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} g^{\Delta_{12}}(t_1, t_2) \Delta_1 \Delta_2 = g(b_1, b_2) - g(a_1, a_2).$$

There is one last point that should be mentioned before moving on. It is possible to think of $[a_1, b_1] \times [a_2, b_2]$ as a measurable subset of $\mathbb{R}^2$, and in so doing, we consider measurable functions, in which case the Lebesgue integral is needed. Just as in the continuous case, the extension of the univariate case to the multivariate case is relatively simple since the measure theory allows the extension in the regular fashion. Thus, just as in the univariate case, all of the standard theorems of Lebesgue theory carry over to the generalized time scales setting (in particular, the theorems hold for the iterated integrals, which is the main focus here).

Lebesgue theory also allows the introduction of the multivariate *improper* integral. Particularly, Lebesgue theory allows for successful evaluation of the improper integral of the first kind given by

$$\int_0^\infty f(t_1, t_2) \Delta_1$$

by considering the measurable set $[0, \infty)$ and the measurable function $f(t_1, t_2)$.

Example 2. Suppose that the value of the double integral

$$\int_2^8 \int_0^5 (2t_1 + 5)(7t_2^2) \Delta_1 \Delta_2$$

for $\mathbb{T}_1 = 5\mathbb{Z}$, and $\mathbb{T}_2 = 2^{\mathbb{Z}} \cup \{0\}$ is needed. Then, using Example 1 and the Fundamental Theorem of Calculus, it follows that

$$\begin{aligned}\int_2^8 \int_0^5 (2t_1+5)(7t_2^2)\Delta_1\Delta_2 &= \int_2^8 (7t_2^2)\left(\int_0^5 (2t_1+5)\Delta_1\right)\Delta_2 \\ &= \int_2^8 (7t_2^2)\left(t_1^2\Big|_{t_1=0}^{t_1=5}\right)\Delta_2 = \left(t_2^3\Big|_{t_2=2}^{t_2=8}\right)\left(t_1^2\Big|_{t_1=0}^{t_1=5}\right) \\ &= (512-8)(25-0) = 12600.\end{aligned}$$

Now that the necessary multivariable calculus has been established, partial dynamic equations and operators can be discussed. The following definition is similar to the one offered by Cheng [4] for the corresponding discrete case.

Definition 4. A *Δ-partial dynamic equation* (Δ-PDE) in the two independent variables t_1 and t_2 on the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively, is a differential equation of the form

$$F(u^{\Delta_1^n}, u^{\Delta_2^n}, u^{\Delta_1^{n\sigma_1}}, u^{\Delta_2^{n\sigma_2}}, u^{\Delta_1^{n-1}\Delta_2}, u^{\Delta_1\Delta_2^{n-1}}, \dots, u^{\sigma_1}, u^{\sigma_2}, u) = g(t_1, t_2).$$

The equation is said to be *linear* if $F(x_1, \dots, x_n)$ is linear, i.e. the equation

$$F(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n) = \alpha F(x_1, \dots, x_n) + \beta F(y_1, \dots, y_n)$$

holds for all $\alpha, \beta \in \mathbb{R}$. The Δ-PDE given above is *homogeneous* if $g(t_1, t_2) = 0$ and is *nonhomogeneous* otherwise. The equation has *order m*, where m is the highest order partial derivative taken with nonzero coefficient in F. The function $f(t_1, t_2) \in C^n(\mathbb{T}_1 \times \mathbb{T}_2)$ is a *solution* of the Δ-PDE if the first equation above is satisfied, i.e., f is a solution if

$$F(f^{\Delta_1^n}, f^{\Delta_2^n}, f^{\Delta_1^{n-1}\Delta_2}, f^{\Delta_1\Delta_2^{n-1}}, \dots, f^{\Delta_1}, f^{\Delta_2}, f) = g(t_1, t_2).$$

Note that in the definition above, F was given in functional form. It is easy to see that F can also be thought of as an operator, defined as follows: $F : C^n(\mathbb{T}_1 \times \mathbb{T}_2) \to C(\mathbb{T}_1 \times \mathbb{T}_2)$ as given in Definition 4. For our purposes, we will mostly be concerned with F being a linear operator. The advantage of thinking of F as an operator is immediately seen in the following theorems:

Theorem 3 (*Principle of Superposition for PDEs*). *Let F be a linear operator, and be as defined in Definition* 4. *If $u_1, u_2, \dots, u_n$ are solutions of $Fu = 0$, then so is $u = c_1u_1 + c_2u_2 + \cdots + c_nu_n$, where $c_1, c_2, \dots, c_n \in \mathbb{R}$ are arbitrary constants.*

Theorem 4. *If F is a linear operator and if $f(t_1, t_2)$ is a solution of $Fu = g$, with F and g as defined in Definition* 4, *and $u_1, u_2, \dots, u_n$ are solutions to $Fu = 0$, then $u = c_1u_1 + c_2u_2 + \cdots + c_nu_n + f$ with $c_1, c_2, \dots, c_n \in \mathbb{R}$ is a solution to $Fu = g$.*

Note that the proofs have been suppressed for brevity's sake. Hereafter, the notation u_h and u_p will denote the *homogeneous* and *particular* solutions of $Fu = g$, respectively. Thus, just as in the continuous case, u_h represents the homogeneous solution found by solving $Fu = 0$ and u_p represents the particular solution found in solving $Fu = g$. These ideas show that linear PDEs on arbitrary time scales are quite similar to the continuous case. To find solutions of a linear PDE of the form $Fu = g$, a solution of the corresponding homogeneous PDE must be found, and then a particular solution must be found. We will see later, however, that there are some complications in general, even for the linear case.

Before moving on to examining solutions of certain operators, it will first be necessary to introduce the Δ-Laplace Transform for the bivariate case.

Definition 5. Let $0 \in \mathbb{T}_1$, $\sup \mathbb{T}_1 = \infty$ and $t_1 \in \mathbb{T}_1$. The Δ-Laplace Transform of the function $f(t_1, t_2)$ (for $f \in C^n(\mathbb{T}_1 \times \mathbb{T}_2)$) with respect to t_1 is given by

$$\mathscr{L}\{f\}(z, t_2) = F(z, t_2) = \int_0^\infty e^{\sigma_1}_{\ominus z}(t_1, 0) f(t_1, t_2) \Delta_1$$

The Δ-Laplace Transform of f with respect to t_2 is defined similarly as long as $\mathbb{T}_2$ has the same form as $\mathbb{T}_1$ does as given in the definition. Note that in the bivariate case, the Laplace Transform is an (improper) iterated integral of t_1. This fact becomes useful because Integration by Parts can be used to develop properties for the bivariate case that are similar to their univariate counterparts. The properties that follow are proven in Bohner and Peterson [2] for the univariate case, and since the proofs are similar for the bivariate case, we omit them here.

Theorem 5. *Assume $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ is such that f^{Δ_1} is continuous. Then*

$$\mathscr{L}\{f^{\Delta_1}\}(z, t_2) = zF(z, t_2) - f(0, t_2)$$

for those regressive $z \in \mathbb{C}$ (with respect to t_1) satisfying

$$\lim_{t_1 \to \infty} \{f(t_1, t_2) e_{\ominus z}(t_1, 0)\} = 0.$$

Theorem 6. *Assume $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ is such that $f^{\Delta_1^i}$ is continuous for all $i = 1, 2, \ldots, n$. Then*

$$\mathscr{L}\{f^{\Delta_1^n}\}(z) = z^n F(z, t_2) - z^{n-1} f(0, t_2) - z^{n-2} f^{\Delta_1}(0, t_2) \cdots - f^{\Delta_1^{n-1}}(0, t_2)$$

for those regressive $z \in \mathbb{C}$ (with respect to t_1) satisfying

$$\lim_{t_1 \to \infty} \{f(t_1, t_2) e_{\ominus z}(t_1, 0)\} = 0.$$

Theorem 7. *Assume $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ is such that $f^{\Delta_2^i}$ is continuous for all $i = 1, 2, \ldots, n$. Then*

$$\mathscr{L}\{f^{\Delta_2^n}\}(z, t_2) = F^{\Delta_2^n}(z, t_2).$$

Proof.

$$\begin{aligned}\mathscr{L}\{f^{\Delta_2^n}\}(z, t_2) &= \int_0^\infty f^{\Delta_2^n}(t_1, t_2) e^{\sigma_1}_{\ominus z}(t_1, 0) \Delta_1 \\ &= \left(\int_0^\infty f(t_1, t_2) e^{\sigma_1}_{\ominus z}(t_1, 0) \Delta_1 \right)^{\Delta_2^n} \\ &= (\mathscr{L}\{f(t_1, t_2)\})^{\Delta_2^n}(z, t_2) = F^{\Delta_2^n}(z, t_2)\end{aligned}$$

where the second statement follows from the Lebesgue Dominated Convergence Theorem.

3. Partial differential operators

We are now in a position where we can examine solutions of PDEs. Focus here will first be placed on the generalizations of solutions to two of the major operators from the continuous and discrete cases: namely the heat and wave operators. Before considering the first of these, distinction must be made between the types of problems that will be encountered in working with PDEs. First, there are *initial value problems* (IVPs), in which initial values for a function and its derivatives are given. Second, there are *initial boundary value problems* (IBVPs), in which initial values and boundary values are specified. Third, there are *boundary value problems* (BVPs) in which only boundary conditions or values are specified. We will examine both homogeneous and nonhomogeneous operators with both homogeneous and nonhomogeneous boundary conditions. As the technique that will be used is the Laplace Transform which requires the use of initial values, BVPs such as Laplace's equation in its usual form will not be considered in this work.

3.1. The homogeneous heat operator

The homogeneous *Heat Equation* on $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$ in one (spatial) dimension has the functional form

$$u^{\Delta_1} = c^2 u^{\Delta_2^2},$$

where $c \in \mathbb{R}$ is constant. It is easy to see that this is a second order linear Δ-PDE. If one prefers operator notation, then define the operator $H : C^2(\mathbb{T}_1 \times \mathbb{T}_2) \to C(\mathbb{T}_1 \times \mathbb{T}_2)$ by $Hu = u^{\Delta_1} - c^2 u^{\Delta_2^2}$, and so a solution to $Hu = 0$ is needed. For purposes of finding at least one solution to this PDE by using the Laplace Transform, we may either impose initial values of u or a combination of initial values for u and boundary conditions for the function. First, consider imposing an initial value of u with respect to t_1 (note that only one is needed since the equation is first order in t_1) and then imposing nonhomogeneous boundary conditions (of which two are needed since the equation is second order in t_2). Thus, a solution of the IBVP

$$\begin{aligned}
&u^{\Delta_1} = c^2 u^{\Delta_2^2}\\
&u(0, t_2) = f(t_2)\\
&\alpha u(t_1, a) - \beta u^{\Delta_2}(t_1, a) = g(t_1), \quad \gamma u(t_1, \sigma_2^2(b)) + \delta u^{\Delta_2}(t_1, \sigma_2(b)) = h(t_1)
\end{aligned}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ is needed. To begin searching for a solution of this IBVP, we take the Laplace Transform with respect to t_1 (we choose to transform in t_1 since our initial value is given in terms of t_1) of both sides of the equation and the boundary conditions to yield the equivalent BVP (note that the initial value problem is solved by using the transform, and so the derivative with respect to t_1 is turned into multiplication by z)

$$\begin{aligned}
&U^{\Delta_2^2}(z, t_2) - \frac{z}{c^2} U(z, t_2) = -\frac{1}{c^2} f(t_2),\\
&\alpha U(z, a) + \beta U^{\Delta_2}(z, a) = G(z), \;\; \gamma U(z, \sigma_2^2(b)) + \delta U^{\Delta_2}(z, \sigma_2(b)) = H(z).
\end{aligned}$$

If we carefully examine the equation above, it should be noted that the transformed equation is an ODE in t_2. If $U_h(z, t_2)$ denotes the homogeneous solution of this ODE, then it follows that the general solution has form

$$U_h(z, t_2) = c_1 e_{\sqrt{z}/c}(t_2, a) + c_2 e_{-\sqrt{z}/c}(t_2, a).$$

We now proceed to solve the BVP. First, note that we must assume that the solution of the corresponding homogeneous BVP with homogeneous boundary conditions has only the trivial solution. Now, solving for c_1 and c_2 yields the matrix equation

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M^{-1} \cdot \begin{pmatrix} G(z) \\ H(z) - Q(z) \end{pmatrix}$$

for

$$M = \begin{pmatrix} \alpha - \frac{\sqrt{z}}{c}\beta & \alpha + \frac{\sqrt{z}}{c}\beta \\ \gamma e_{\sqrt{z}/c}(\sigma_2^2(b), a) + \frac{\sqrt{z}}{c}\delta e_{\sqrt{z}/c}(\sigma_2(b), a) & \gamma e_{-\sqrt{z}/c}(\sigma_2^2(b), a) - \frac{\sqrt{z}}{c}\delta e_{-\sqrt{z}/c}(\sigma_2(b), a) \end{pmatrix}$$

and

$$\begin{aligned} Q(z) &= \gamma e_{-\sqrt{z}/c}(\sigma_2^2(b), a) \cdot \int_a^{\sigma_2^2(b)} \frac{f(\tau)}{2c\sqrt{z}e^{\sigma_2}_{-\sqrt{z}/c}(\tau, a)}\,\Delta\tau \\ &\quad + \gamma e_{\sqrt{z}/c}(\sigma_2^2(b), a) \cdot \int_a^{\sigma_2^2(b)} \frac{f(\tau)}{2c\sqrt{z}e^{\sigma_2}_{\sqrt{z}/c}(\tau, a)}\,\Delta\tau \\ &\quad - \frac{\sqrt{z}}{c}\,\delta e_{\sqrt{z}/c}(\sigma_2(b), a) \cdot \int_a^{\sigma_2(b)} \frac{f(\tau)}{2c\sqrt{z}e^{\sigma_2}_{\sqrt{z}/c}(\tau, a)}\,\Delta\tau \\ &\quad - \frac{\sqrt{z}}{c}\,\delta e_{-\sqrt{z}/c}(\sigma_2(b), a) \cdot \int_a^{\sigma_2(b)} \frac{f(\tau)}{2c\sqrt{z}e^{\sigma_2}_{-\sqrt{z}/c}(\tau, a)}\,\Delta\tau. \end{aligned}$$

The validity of this last statement follows from the fact that the variation of parameters formula gives the solution of the nonhomogeneous ODE. In this spirit, first note that the Wronskian of the two homogeneous solutions is

$$\begin{aligned} W(e_{\sqrt{z}/c}, e_{-\sqrt{z}/c}) &= \begin{vmatrix} e_{\sqrt{z}/c} & e_{-\sqrt{z}/c} \\ \frac{\sqrt{z}}{c}e_{\sqrt{z}/c} & -\frac{\sqrt{z}}{c}e_{-\sqrt{z}/c} \end{vmatrix} \\ &= -\frac{\sqrt{z}}{c}\,e_{\sqrt{z}/c}e_{-\sqrt{z}/c} - \frac{\sqrt{z}}{c}\,e_{\sqrt{z}/c}e_{-\sqrt{z}/c} = -2\,\frac{\sqrt{z}}{c}\,e_{\sqrt{z}/c}e_{-\sqrt{z}/c}. \end{aligned}$$

Then the variation of parameters formula yields

$$U_p(z, t_2) = \frac{1}{2c\sqrt{z}} \int_a^{t_2} \frac{e^{\sigma_2}_{\sqrt{z}/c}(\tau, a)e_{-\sqrt{z}/c}(t_2, a) - e^{\sigma_2}_{-\sqrt{z}/c}(\tau, a)e_{\sqrt{z}/c}(t_2, a)}{e^{\sigma_2}_{\sqrt{z}/c}(\tau, a)e_{-\sqrt{z}/c}(\tau, a)} f(\tau)\,\Delta\tau.$$

The solution of the ODE is then given by $U(z, t_2) = c_1 e_{\sqrt{z}/c}(t_2, a) + c_2 e_{-\sqrt{z}/c}(t_2, a) + U_p(z, t_2)$.

It is easy to see that at this point that without an inverse Laplace Transform, there is little hope of finding a solution of the PDE in general. However, it is possible in certain cases to obtain solutions. For example, if we assume homogeneous boundary solutions, i.e., $G(z) = H(z) = 0$, or simply no boundary conditions (so that we are trying to solve an IVP in this case), then our problem becomes much simpler. In fact, assuming these conditions in combination with elementary choices for $f(t_2)$ gives rise to solutions by using the

Undetermined Coefficients technique. We shall illustrate the idea with the generalized polynomials (the reader can see the Appendix in Section 5 for the definition of the generalized polynomials); the results in the table that follows the example can be obtained by similar calculations. Note that solutions involving any linear combination of elementary choices for $f(t_2)$ can be found since the equation is linear.

Example 3. Consider the IVP

$$\begin{cases} u^{\Delta_1} = c^2 u^{\Delta_2^2} \\ u(0, t_2) = h_k(t_2, 0). \end{cases}$$

Preceding discussion shows that the IVP is equivalent to the ODE

$$U^{\Delta_2^2} - \frac{z}{c^2} U = -\frac{1}{c^2} h_k(t_2, 0).$$

Undetermined Coefficients is employed to find the particular solution. Thus, assume that the particular solution has the form

$$U_p(z, t_2) = a_1 h_k(t_2, 0) + a_2 h_{k-1}(t_2, 0) + \cdots + a_n h_0(t_2, 0)$$

where $a_1, a_2, \ldots, a_n$ are constants to be determined. If $U_p(z, t_2)$ is of this form, then

$$U_p^{\Delta_2^2}(z, t_2) = a_1 h_{k-2}(t_2, 0) + a_2 h_{k-3}(t_2, 0) + \cdots + a_{n-2} h_0(t_2, 0).$$

Substituting these values into the differential equation yields the algebraic equation

$$\begin{aligned} &(a_1 h_{k-2}(t_2, 0) + a_2 h_{k-3}(t_2, 0) + \cdots + a_{n-2} h_0(t_2, 0)) \\ &\qquad - \frac{z}{c^2}(a_1 h_k(t_2, 0) + a_2 h_{k-1}(t_2, 0) + \cdots + a_n h_0(t_2, 0)) = -\frac{1}{c^2} h_k(t_2, 0). \end{aligned}$$

Close examination of this equation immediately produces the values $a_1 = 1/z$ and $a_2 = 0$. The remaining a_i are a bit more complicated to find. However, upon careful inspection of the equation, the recursive relations $a_{2i+1} - (z/c^2) a_{2i+3} = 0$ and $a_{2i} - a_{2i+2} = 0$ can be seen to hold for $i = 0, 1, 2, \ldots$. Using the fact that $a_2 = 0$, it is easy to see that a_{2i} is zero for all i. Then, using the fact that $a_1 = 1/z$, it follows that $a_{2i+1} = c^{2i}/z^{i+1}$. This information will then lead to a solution, for then

$$U_p(z, t_2) = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{c^{2j}}{z^{j+1}} h_{k-2j}(t_2, 0),$$

where $\lfloor k/2 \rfloor$ denotes the floor of $k/2$, and so

$$u_p(t_1, t_2) = \sum_{j=0}^{\lfloor k/2 \rfloor} c^{2j} h_j(t_1, 0) h_{k-2j}(t_2, 0).$$

We now verify that the function given actually solves the IVP. Thus, first note that

$$u_p(0, t_2) = \sum_{j=0}^{\lfloor k/2 \rfloor} c^{2j} h_j(0, 0) h_{k-2j}(t_2, 0) = (1)(h_k(t_2, 0)) = h_k(t_2, 0),$$

since $h_0(t,s) \equiv 1$ for all t, s and $h_j(0,0) \equiv 0$ for $j > 0$, and so the function given satisfies the initial condition. Second, if we adopt the standard convention that negative subscripts in an expression vanish, then it follows that

$$\begin{aligned} u_p^{\Delta_1} &= \left(\sum_{j=0}^{\lfloor k/2 \rfloor} c^{2j} h_j(t_1,0) h_{k-2j}(t_2,0) \right)^{\Delta_1} \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor} c^{2j} h_{j-1}(t_1,0) h_{k-2j}(t_2,0) = \sum_{j=0}^{\lfloor k/2 \rfloor -1} c^{2j+2} h_j(t_1,0) h_{k-2(j+1)}(t_2,0), \end{aligned}$$

since $j-1$ is a negative subscript when $j=0$. Likewise,

$$\begin{aligned} u_p^{\Delta_2^2} &= \left(\sum_{j=0}^{\lfloor k/2 \rfloor} c^{2j} h_j(t_1,0) h_{k-2j}(t_2,0) \right)^{\Delta_2^2} \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor} c^{2j} h_j(t_1,0) h_{k-2(j+1)}(t_2,0) = \sum_{j=0}^{\lfloor k/2 \rfloor -1} c^{2j} h_j(t_1,0) h_{k-2(j+1)}(t_2,0), \end{aligned}$$

where the last equality sign holds since $k-2(j+1)$ is a negative subscript when $j = \lfloor k/2 \rfloor$. Therefore, it is indeed clearly the case that

$$u_p^{\Delta_1} = c^2 u_p^{\Delta_2^2},$$

and so the function is a solution.

Although at this point we may not be able to find the general solution of the IBVP with arbitrary boundary conditions, there is one important thing that we can note. If we examine the nature of solutions given by the transformed equation, then assuming an inverse exists, we expect that the general solution would have form

$$u(t_1,t_2) = P *_1 g + Q *_1 h + R *_2 f,$$

where P, Q, R represent the corresponding inverse transforms of the coefficients of G, H, and f, respectively, and $*_1$ denotes convolution with respect to t_1 and $*_2$ denotes convolution with respect to t_2. Thus, we see that the arbitrary time scale is comparable in solutions to the continuous case (Table 1).

3.2. The homogeneous wave operator

The homogeneous *Wave Equation* on $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$ in one (spatial) dimension has the functional form

$$u^{\Delta_1^2} = c^2 u^{\Delta_2^2}$$

where $c \in \mathbb{R}$ is constant. Clearly, the wave operator is another second order linear Δ-PDE. If one prefers operator notation, then define the operator $W : C^2(\mathbb{T}_1 \times \mathbb{T}_2) \to C(\mathbb{T}_1 \times \mathbb{T}_2)$ by $Wu = u^{\Delta_1^2} - c^2 u^{\Delta_2^2}$, and so a solution to $Wu = 0$ is needed. Invoking the use of the Laplace Transform requires us to impose initial

Table 1
Particular solutions to $u^{\Delta_1} = c^2 u^{\Delta_2^2}$ with $u(0, t_2) = f(t_2)$

$f(t_2)$	$u_p(t_1, t_2)$
$h_k(t_2, 0)$	$\sum_{j=0}^{\lfloor k/2 \rfloor} c^{2j} h_j(t_1, 0) h_{k-2j}(t_2, 0)$
$e_q(t_2, 0)$	$e_{c^2q^2}(t_1, 0) e_q(t_2, 0)$
$\cos_q(t_2, 0)$	$e_{-c^2q^2}(t_1, 0) \cos_q(t_2, 0)$
$\sin_q(t_2, 0)$	$e_{-c^2q^2}(t_1, 0) \sin_q(t_2, 0)$
$\cosh_q(t_2, 0)$	$e_{c^2q^2}(t_1, 0) \cosh_q(t_2, 0)$
$\sinh_q(t_2, 0)$	$e_{c^2q^2}(t_1, 0) \sinh_q(t_2, 0)$

values of u with respect to t_1. We may also impose boundary conditions in terms of t_2 if we so choose. Thus, a solution of the IBVP

$$\begin{aligned} &u^{\Delta_1^2} = c^2 u^{\Delta_2^2}, \\ &u(0, t_2) = f(t_2), \quad u^{\Delta_1}(0, t_2) = g(t_2), \\ &\alpha u(t_1, a) - \beta u^{\Delta_2}(t_1, a) = h(t_1), \quad \gamma u(t_1, \sigma_2^2(b)) + \delta u^{\Delta_2}(t_1, \sigma_2(b)) = j(t_1), \end{aligned}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ is desired. Transforming in t_1 yields the BVP

$$\begin{aligned} &U^{\Delta_2^2}(z, t_2) - \frac{z^2}{c^2} U(z, t_2) = -\frac{z}{c^2} f(t_2) - \frac{1}{c^2} g(t_2), \\ &\alpha U(z, a) + \beta U^{\Delta_2}(z, a) = H(z), \quad \gamma U(z, \sigma_2^2(b)) + \delta U^{\Delta_2}(z, \sigma_2(b)) = J(z). \end{aligned}$$

Thus, just as in the case of the heat equation, we arrive at an ODE in t_2. If $U_h(z, t_2)$ denotes the homogeneous solution of this ODE, then it follows that the general solution has form

$$U_h(z, t_2) = c_1 e_{z/c}(t_2, a) + c_2 e_{-z/c}(t_2, a).$$

Again, we assume that the trivial solution is the sole solution to the homogeneous BVP with homogeneous boundary conditions. Applying the boundary conditions and solving for c_1 and c_2 yields the matrix equation

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M^{-1} \cdot \begin{pmatrix} H(z) \\ J(z) - Q(z) \end{pmatrix}$$

with

$$M = \begin{pmatrix} \alpha - \frac{z}{c}\beta & \alpha + \frac{z}{c}\beta \\ \gamma e_{z/c}(\sigma_2^2(b), a) + \frac{z}{c}\delta e_{z/c}(\sigma_2(b), a) & \gamma e_{-z/c}(\sigma_2^2(b), a) - \frac{z}{c}\delta e_{-z/c}(\sigma_2(b), a) \end{pmatrix}$$

and

$$\begin{aligned} Q(z) = {} & \gamma e_{-z/c}(\sigma_2^2(b), a) \int_a^{\sigma_2^2(b)} \frac{f(\tau) + \frac{1}{z} g(\tau)}{2c e_{-z/c}^{\sigma_2}(\tau, a)} \Delta\tau + \gamma e_{z/c}(\sigma_2^2(b), a) \int_a^{\sigma_2^2(b)} \frac{f(\tau) + \frac{1}{z} g(\tau)}{2c e_{z/c}^{\sigma_2}(\tau, a)} \Delta\tau \\ & - \frac{z}{c}\delta e_{z/c}(\sigma_2(b), a) \int_a^{\sigma_2(b)} \frac{f(\tau) + \frac{1}{z} g(\tau)}{2c e_{z/c}^{\sigma_2}(\tau, a)} \Delta\tau - \frac{z}{c}\delta e_{-z/c}(\sigma_2(b), a) \int_a^{\sigma_2(b)} \frac{f(\tau) + \frac{1}{z} g(\tau)}{2c e_{-z/c}^{\sigma_2}(\tau, a)} \Delta\tau. \end{aligned}$$

The variation of parameters formula was used again to find the solution of the nonhomogeneous ODE which was necessary to compute c_1 and c_2. To verify this solution, we need only note that the Wronskian of the two homogeneous solutions is

$$W(e_{z/c}, e_{-z/c}) = \begin{vmatrix} e_{z/c} & e_{-z/c} \\ \frac{z}{c}\, e_{z/c} & -\frac{z}{c}\, e_{-z/c} \end{vmatrix} = -\frac{z}{c}\, e_{-z/c}e_{z/c} - \frac{z}{c}\, e_{-z/c}e_{z/c} = -2\frac{z}{c}\, e_{z/c}e_{-z/c}.$$

Thus, the variation of parameters formula gives

$$U_p(z, t_2) = \frac{1}{2c}\int_a^{t_2} \frac{e^{\sigma_2}_{z/c}(\tau, a)e_{-z/c}(t_2, a) - e^{\sigma_2}_{-z/c}(\tau, a)e_{z/c}(t_2, a)}{e^{\sigma_2}_{z/c}(\tau, a)e_{-z/c}(\tau, a)} \left(f(\tau) + \frac{1}{z}\, g(\tau) \right) \Delta\tau,$$

so that $U(z, t_2) = c_1 e_{z/c}(t_2, a) + c_2 e_{-z/c}(t_2, a) + U_p(z, t_2)$.

Once again, with no inversion formula for the transform, at this point a general solution is not tractable. However, solutions to the IVP are possible when f and g are one of the six elementary functions mentioned earlier by using the Undetermined Coefficients technique. The table of solutions for the wave equation when f and g are one of these functions follows this discussion.

Note that if $f = f_1 + f_2$ or $g = g_1 + g_2$, for $f_1,\ f_2,\ g_1,\ g_2$ any linear combination of the six functions, then solutions can be found using the table above by simply adding the corresponding single solutions to $f_1,\ f_2,\ g_1,\ g_2$. Note that the table can also be used to "mix" the functions f and g, i.e., f and g can be different functions rather than the same as they are in the table. For example, suppose that $f(t_2) = \cos_q(t_2, 0) + \sin_m(t_2, 0)$ and $g(t_2) = h_r(t_2, 0)$. Then a solution to

$$u^{\Delta_1^2} = c^2 u^{\Delta_2^2},$$
$$u(0, t_2) = f(t_2), \quad u^{\Delta_1}(0, t_2) = g(t_2)$$

in this case according to the table is

$$\begin{aligned} u_p(t_1, t_2) &= u_{p_q}(t_1, t_2) + u_{p_m}(t_1, t_2) + u_{p_r}(t_1, t_2) \\ &= \cos_{cq}(t_1, 0)\cos_q(t_2, 0) + \cos_{cm}(t_1, 0)\sin_m(t_2, 0) + \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} c^{2i} h_{2i+1}(t_1, 0)h_{r-2i}(t_2, 0). \end{aligned}$$

Finally, just as in the heat equation, it is worth noting that solutions in general to the wave equation are similar to the continuous case in that our discussion of the transform of the equation leads to the conclusion that solutions are of the form $P *_1 f + Q *_1 g + R *_2 h + S *_2 j$, where the notation is as given in the discussion of the heat equation (Table 2).

4. Nonhomogeneous operators

Thus far, we have examined the homogeneous heat and wave operators with corresponding operator notations $Hu = 0$ and $Wu = 0$. Attention is now turned to the *nonhomogeneous* equations with operator notations $Hu = f$ and $Wu = f$. The technique for solving nonhomogeneous PDEs is the same as the

Table 2
Particular solutions to $u^{\Delta_1^2} = c^2 u^{\Delta_2^2}$ with $u(0, t_2) = f(t_2)$ and $u^{\Delta_1}(0, t_2) = g(t_2)$

$f(t_2)$	$g(t_2)$	$u_{p_q}(t_1, t_2)$	$u_{p_r}(t_1, t_2)$
$e_q(t_2, 0)$	$e_r(t_2, 0)$	$\cosh_{cq}(t_1, 0)e_q(t_2, 0)$	$\frac{1}{cr}\sinh_{cr}(t_1, 0)e_r(t_2, 0)$
$h_q(t_2, 0)$	$h_r(t_2, 0)$	$\sum_{i=0}^{\lfloor \frac{q}{2} \rfloor} c^{2i} h_{2i}(t_1, 0)h_{q-2i}(t_2, 0)$	$\sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} c^{2i} h_{2i+1}(t_1, 0)h_{r-2i}(t_2, 0)$
$\cos_q(t_2, 0)$	$\cos_r(t_2, 0)$	$\cos_{cq}(t_1, 0)\cos_q(t_2, 0)$	$\frac{1}{cr}\sin_{cr}(t_1, 0)\cos_r(t_2, 0)$
$\sin_q(t_2, 0)$	$\sin_r(t_2, 0)$	$\cos_{cq}(t_1, 0)\sin_q(t_2, 0)$	$\frac{1}{cr}\sin_{cr}(t_1, 0)\sin_r(t_2, 0)$
$\cosh_q(t_2, 0)$	$\cosh_r(t_2, 0)$	$\cosh_{cq}(t_1, 0)\cosh_q(t_2, 0)$	$\frac{1}{cr}\sinh_{cr}(t_1, 0)\cosh_r(t_2, 0)$
$\sinh_q(t_2, 0)$	$\sinh_r(t_2, 0)$	$\cosh_{cq}(t_1, 0)\sinh_q(t_2, 0)$	$\frac{1}{cr}\sinh_{cr}(t_1, 0)\sinh_r(t_2, 0)$

technique for solving nonhomogeneous ODEs: first find the corresponding homogeneous solutions and then find a particular solution to the nonhomogeneous equation. We have already seen that at this point, the homogeneous solution is not attainable by the Laplace Transform since no formula for the inverse is currently known. However, just as in the homogeneous case, a particular solution is tractable when *f* is *separable*, i.e. when $f(t_1, t_2) = g(t_1) + h(t_2)$ or $f(t_1, t_2) = g(t_1)h(t_2)$, and when g and h are one of the six elementary functions mentioned earlier. The effect of *f* on the solutions is that the integrals for the particular solutions obtained by variation of parameters in the corresponding solutions to the transformed nonhomogeneous ODEs must be modified by *f*. Thus, first recall that the particular solution of the homogeneous heat equation $Hu = 0$ with $u(0, t_2) = g(t_2)$ is

$$U_p(z, t_2) = \frac{1}{2c\sqrt{z}} \int_a^{t_2} \frac{e^{\sigma_2}_{\sqrt{z}/c}(\tau, a)e_{-\sqrt{z}/c}(t_2, a) - e^{\sigma_2}_{-\sqrt{z}/c}(\tau, a)e_{\sqrt{z}/c}(t_2, a)}{e^{\sigma_2}_{\sqrt{z}/c}(\tau, a)e_{-\sqrt{z}/c}(\tau, a)} g(\tau)\Delta\tau,$$

and so the particular solution to the nonhomogeneous equation $Hu = f$ with $u(0, t_2) = g(t_2)$ is

$$U_p(z, t_2) = \frac{1}{2c\sqrt{z}} \int_a^{t_2} \frac{e^{\sigma_2}_{\sqrt{z}/c}(\tau, a)e_{-\sqrt{z}/c}(t_2, a) - e^{\sigma_2}_{-\sqrt{z}/c}(\tau, a)e_{\sqrt{z}/c}(t_2, a)}{e^{\sigma_2}_{\sqrt{z}/c}(\tau, a)e_{-\sqrt{z}/c}(\tau, a)} (f(z, \tau) + g(\tau))\Delta\tau,$$

by variation of parameters. Next, it was shown earlier that the particular solution of the homogeneous wave equation $Wu = 0$ with $u(0, t_2) = g(t_2)$ and $u^{\Delta_1}(0, t_2) = h(t_2)$ is

$$U_p(z, t_2) = \frac{1}{2c} \int_a^{t_2} \frac{e^{\sigma_2}_{z/c}(\tau, a)e_{-z/c}(t_2, a) - e^{\sigma_2}_{-z/c}(\tau, a)e_{z/c}(t_2, a)}{e^{\sigma_2}_{z/c}(\tau, a)e_{-z/c}(\tau, a)} \left(g(\tau) + \frac{1}{z}h(\tau)\right)\Delta\tau,$$

and so the variation of parameters formula applied to the nonohomogenous wave equation $Wu = f$ with the same initial conditions as before yields that

$$U_p(z, t_2) = \frac{1}{2c} \int_a^{t_2} \frac{e^{\sigma_2}_{z/c}(\tau, a)e_{-z/c}(t_2, a) - e^{\sigma_2}_{-z/c}(\tau, a)e_{z/c}(t_2, a)}{e^{\sigma_2}_{z/c}(\tau, a)e_{-z/c}(\tau, a)} \left(g(\tau) + \frac{1}{z}h(\tau) + \frac{1}{z}f(z, \tau)\right)\Delta\tau.$$

We saw that even in the homogeneous case, evaluating these integrals in most cases is unproductive because the inverse transform is still not known. However, if *f*, *g*, and *h* are one of the six elementary functions discussed earlier, then we can use undetermined coefficients to determine solutions. As *f* can become rather complicated even when one just considers linear combinations of these six functions, we give an example of the nonhomogeneous heat equation.

Example 4. Now consider the IVP

$$\begin{cases} u^{\Delta_1^2} - c^2 u^{\Delta_2^2} = \sin_q(t_1,0)\cos_r(t_2,0) \\ u(0,t_2) = h_k(t_2,0) \quad u^{\Delta_1}(0,t_2) = e_m(t_2,0). \end{cases}$$

Transforming the system yields the ODE

$$U^{\Delta_2^2} - \frac{z^2}{c^2} U = \frac{q}{c^2(z^2+q^2)} \cos_r(t_2,0) + \frac{z}{c^2} h_k(t_2,0) + \frac{1}{c^2} e_m(t_2,0).$$

We must only solve the equation

$$U^{\Delta_2^2} - \frac{z^2}{c^2} U = \frac{q}{c^2(z^2+q^2)} \cos_r(t_2,0)$$

since the second part of the particular solution follows from using Table 2. Once again, to solve the ODE above, we employ undetermined coefficients. Thus, a solution of the form $U_{p_r}(z,t_2) = A\cos_r(t_2,0)$ is found. Taking derivatives, substituting appropriate values, and equating coefficients yields

$$U_{p_r}(z,t_2) = -\frac{q}{(z^2+q^2)(z^2+c^2r^2)} \cos_r(t_2,0)$$

so that

$$\begin{aligned} u_{p_r}(t_1,t_2) &= -\left(\sin_q(t_1,0) * \frac{1}{cr}\sin_{cr}(t_1,0)\right)\cos_r(t_2,0) \\ &= \left(\frac{1}{q^2-c^2r^2}\sin_q(t_1,0) - \frac{q}{crq^2-c^3r^3}\sin_{cr}(t_1,0)\right)\cos_r(t_2,0), \end{aligned}$$

where the last equality follows from Bohner and Peterson [2]. Summing all particular solutions gives

$$\begin{aligned} u_p(t_1,t_2) &= u_{p_r}(t_1,t_2) + u_{p_k}(t_1,t_2) + u_{p_m}(t_1,t_2) \\ &= \left(\frac{1}{q^2-c^2r^2}\sin_q(t_1,0) - \frac{q}{crq^2-c^3r^3}\sin_{cr}(t_1,0)\right)\cos_r(t_2,0) \\ &\quad + \sum_{i=0}^{\lfloor k/2 \rfloor} c^{2i} h_{2i}(t_1,0) h_{k-2i}(t_2,0) + \frac{1}{cm}\sinh_{cm}(t_1,0) e_m(t_2,0). \end{aligned}$$

5. Mixed time scales

Up to this point, we have presented solutions on arbitrary time scales, i.e. on $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$. We now wish to examine solutions on some specific time scales a little more closely. As will be seen, solutions

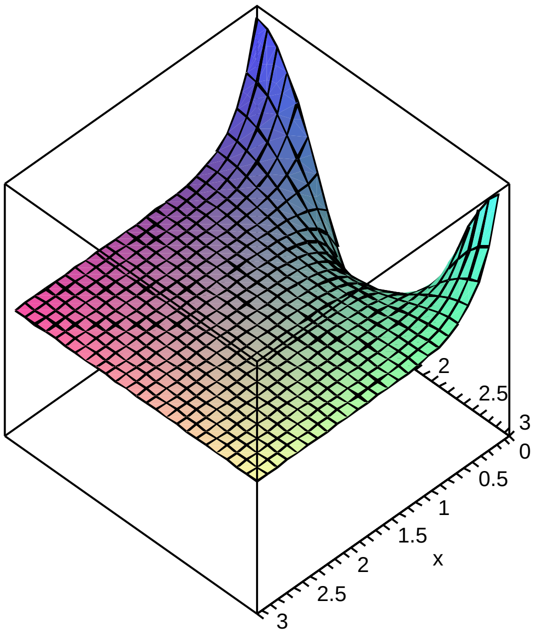

Fig. 1. $\mathbb{R} \times \mathbb{R}$.

can vary greatly depending on the combination of time scales chosen. The classical cases of $\mathbb{R} \times \mathbb{R}$ and $\mathbb{Z} \times \mathbb{Z}$ are of course well understood and remain the center of focus, and so attention here will be devoted to solutions in which the time scales are "mixed", i.e. not the same. As was mentioned in the introduction, we believe that probably the most important application of this work is in numerical analysis since we now have an alternative to adaptive methods. With this in mind, we shall work with mixtures of the time scales $\mathbb{R}$, $h\mathbb{Z}$, and $q^{\mathbb{N}_0}$.

For example, consider the IVP

$$u^{\Delta_1} = u^{\Delta_2^2},$$
$$u(0, t_2) = \cos_2(t_2, 0),$$

where $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$ and $\mathbb{T}_1$ and $\mathbb{T}_2$ are any of $\mathbb{R}$, $0.1\mathbb{Z}$, or $1.1^{\mathbb{N}_0} \cup \{0\}$. We have already seen that according to Table 1, a solution of this PDE for arbitrary $\mathbb{T}_1$ and $\mathbb{T}_2$ is given by

$$u(t_1, t_2) = e_{-4}(t_1, 0)\cos_2(t_2, 0).$$

Thus, for any specific time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ involved, we only need to determine the exponential function corresponding to $\mathbb{T}_1$ and the corresponding cosine function for $\mathbb{T}_2$. The exponential functions of $\mathbb{R}$, $0.1\mathbb{Z}$, and $1.1^{\mathbb{N}_0} \cup \{0\}$ are e^{-4t}, $(0.6)^{t/3}$, and $\prod_{s\in(0,t)}(1 - 0.4s)$, respectively. With these

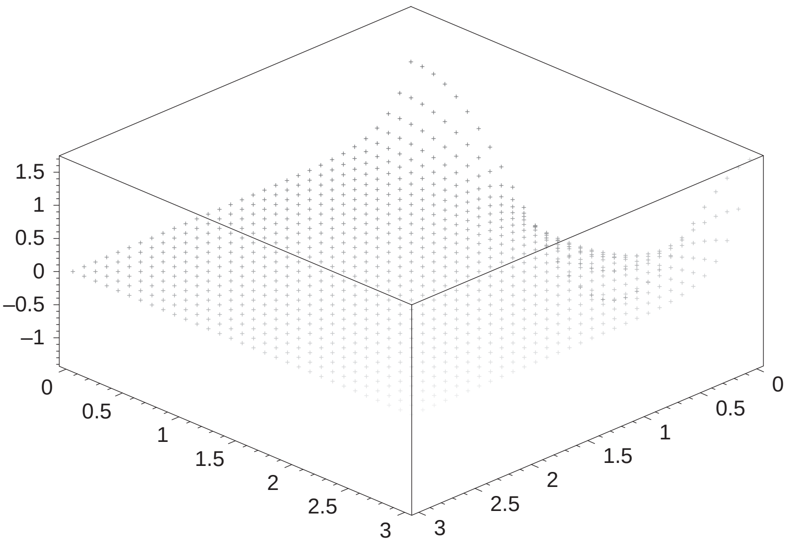

Fig. 2. $0.1\mathbb{Z} \times 0.1\mathbb{Z}$.

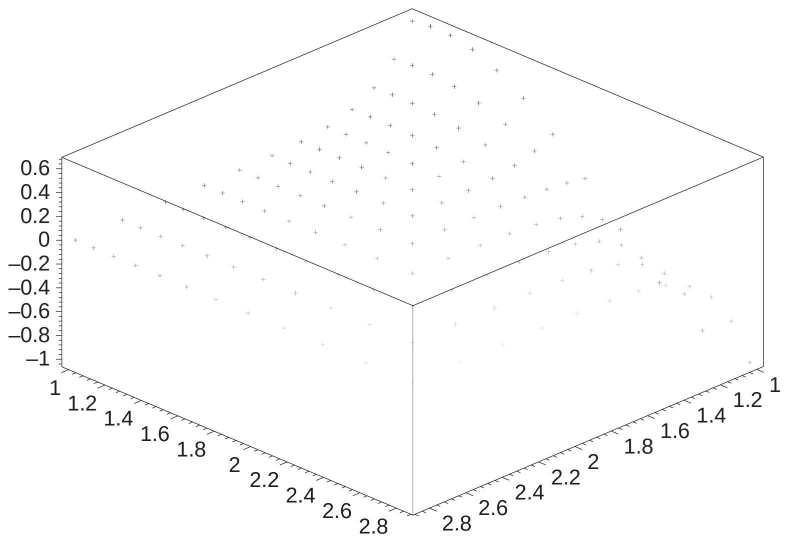

Fig. 3. $1.1^{\mathbb{N}_0} \cup \{0\} \times 1.1^{\mathbb{N}_0} \cup \{0\}$.

exponentials, it follows that the respective corresponding cosine functions are $\cos 2t$, $(1+0.2i)^{10t} + (1-0.2i)^{10t}/2$, and $\prod_{s\in(0,t)}(1+0.2is) + \prod_{s\in(0,t)}(1-0.2is)/2$. With these ideas, the following graphs are offered to illustrate the differences in solutions among the different combinations of the three sets (Figs. 1–5).)

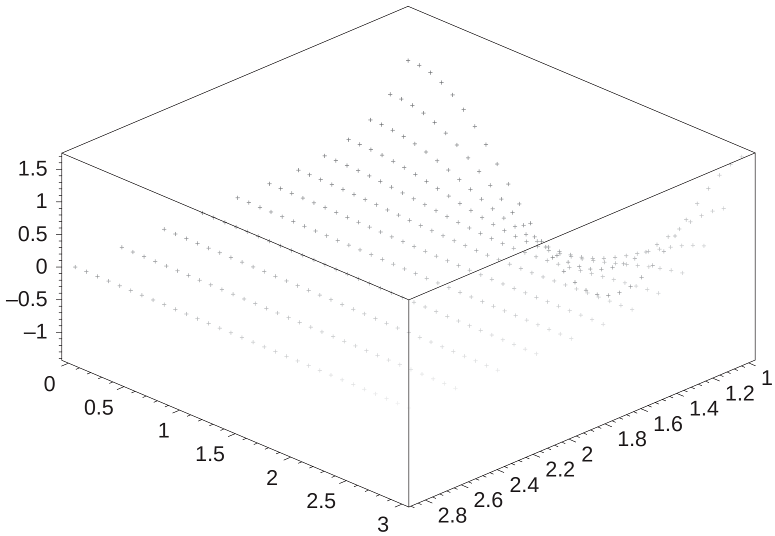

Fig. 4. $1.1^{\mathbb{N}_0} \cup \{0\} \times 0.1\mathbb{Z}$.

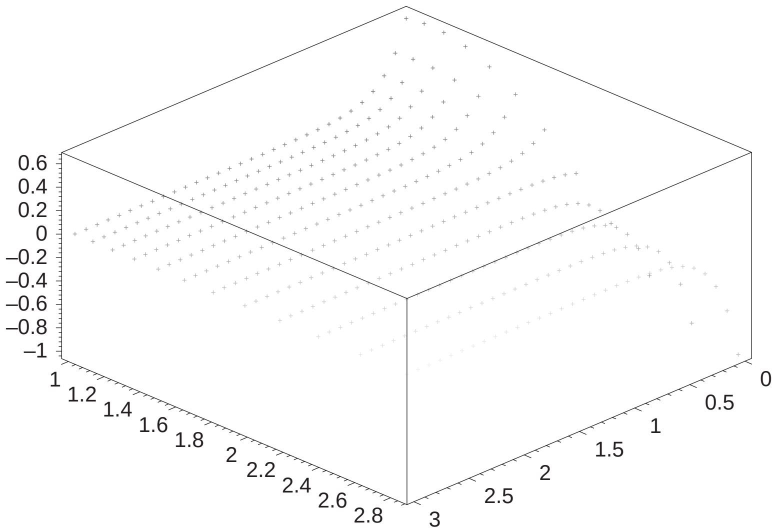

Fig. 5. $0.1\mathbb{Z} \times 1.1^{\mathbb{N}_0} \cup \{0\}$.

Appendix

A time scale is a nonempty closed subset of the reals in the standard topology. Thus, for example, $\mathbb{R}$, $\overline{q^{\mathbb{Z}}}$, and $h\mathbb{Z}$ for $q > 0$ and $h > 0$ both constants are all time scales. Given a time scale $\mathbb{T}$, we define the following: the forward jump operator

$$\sigma : \mathbb{T} \to \mathbb{T}$$

by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator

$$\rho : \mathbb{T} \to \mathbb{T}$$

by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In our definition, we set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, so that $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t and $\rho(t) = t$ if $\mathbb{T}$ has a minimum t. If $\sigma(t) > t$, then we say that t is *right-scattered*, and if $\sigma(t) = t$ we say t is *right-dense*. Likewise, if $\rho(t) < t$, then t is *left-scattered*, and if $\rho(t) = t$, then t is *left-dense*. If t is simultaneously left- and right-scattered, then we say that t is *isolated*.

We also make use of the *graininess function* μ and the set $\mathbb{T}^\kappa$. Each is defined as follows:

$$\mu(t) := \sigma(t) - t$$

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \text{ and left-scattered} \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

Example 5. Consider the time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$, and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

(1) For $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and likewise

$$\rho(t) = \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t.$$

Thus, every point in $\mathbb{R}$ is dense. We also have that

$$\mu(t) = \sigma(t) - t = t - t = 0.$$

(2) For $\mathbb{T} = h\mathbb{Z}$, we have

$$\begin{aligned} \sigma(t) &= \inf\{s \in h\mathbb{Z} : s > t\} \\ &= \inf\{t + h, t + 2h, t + 3h, \ldots\} = t + h \end{aligned}$$

and likewise

$$\rho(t) = \sup\{s \in h\mathbb{Z} : s < t\} = \sup\{t - h, t - 2h, t - 3h, \ldots\} = t - h.$$

From these statements we see that every point in $h\mathbb{Z}$ is isolated, and that

$$\mu(t) = \sigma(t) - t = t + h - t = h.$$

(3) For $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, we have

$$\sigma(t) = \inf\{s \in \overline{q^{\mathbb{Z}}} : s > t\}.$$

Now, if $t \in \mathbb{T}$, then $t = q^n$ or $t = 0$ for $n \in \mathbb{Z}$. Thus,

$$\inf\{s \in \overline{q^{\mathbb{Z}}} : s > t\} = \inf\{q^r : r \in [n+1, \infty)\} = q^{n+1} = qt,$$

and

$$\rho(t) = \sup\{s \in \overline{q^{\mathbb{Z}}} : s < t\} = \sup\{q^r : r \in (-\infty, r-1]\} = q^{r-1} = \frac{t}{q}.$$

Thus, $t = 0$ is right dense and every other $t \in \mathbb{T}$ is isolated. We also have

$$\mu(t) = \sigma(t) - t = qt - t = (q-1)t.$$

Next, we need the *delta derivative* of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^\kappa$:

Definition 6. Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t, with $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for $\delta > 0$ such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s| \text{ for all } s \in U.$$

We call f^Δ the *delta derivative* of f at t.

We say that f is differentiable on $\mathbb{T}^\kappa$ provided the derivative exists for all $t \in \mathbb{T}^\kappa$. Likewise, we call f^Δ the delta derivative of f on $\mathbb{T}^\kappa$.

Bohner and Peterson [2] show the following:

Theorem 8. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function with $t \in \mathbb{T}^\kappa$. Then the following hold*:

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If t is right dense, then f is differentiable at t iff the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

Example 6.

(1) Let $\mathbb{T}=\mathbb{R}$. Then Theorem 8 part (iii) tells us that $f:\mathbb{R}\to\mathbb{R}$ is delta differentiable at $t\in\mathbb{R}$, if and only if

$$f'(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}\text{ exists,}$$

and in which case we have

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}=f'(t)$$

Thus, the delta derivative is just the usual derivative in the continuous case.

(2) Let $\mathbb{T}=h\mathbb{Z}$. Then Theorem 8 part (ii) tells us that $f:h\mathbb{Z}\to\mathbb{R}$ is delta differentiable at $t\in h\mathbb{Z}$ with

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}=\frac{f(t+h)-f(t)}{h}=\Delta f(t)$$

where Δ is the usual *forward difference operator* defined for the discrete case.

Example 7.

(1) Let $f(t)=t^2$. We wish to find the delta derivative of f for the time scale $\mathbb{T}=h\mathbb{Z}$. Using Example 6, we know that

$$f^{\Delta}(t)=\frac{f(t+h)-f(t)}{h}=\frac{(t^2+2th+h^2)-t^2}{h}=2t+h.$$

(2) Let $f(t)=t^3$. Again, we wish to find the delta derivative of f, but this time for the time scale $\mathbb{T}=\overline{q^{\mathbb{Z}}}$. Let $t\neq 0$. Then from Theorem 8 part (ii) and Example 5, we know that

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}=\frac{q^3t^3-t^3}{(q-1)t}=t^2(q^2+q+1)$$

If $t=0$ (which is right-dense as we saw in Example 5), then Theorem 8 part(iii) gives us that

$$f^{\Delta}(0)=\lim_{s\to 0}\frac{f(0)-f(s)}{0-s}=\lim_{s\to 0}s^2=0.$$

Thus, in either case, the equation $f^{\Delta}(t)=t^2(q^2+q+1)$ holds.

(3) Let $\mathbb{T}$ be an arbitrary time scale and let $f(t)=c$, where c is a constant. We claim that $f^{\Delta}(t)=0$. To see this, note that $f(\sigma(t))=f(t)=c$ for all t, so that given $\varepsilon>0$ we have

$$|f(\sigma(t))-f(s)-0\cdot[\sigma(t)-s]|=|c-c|=0\leqslant\varepsilon|\sigma(t)-s|$$

which hold *for all* $s\in\mathbb{T}$.

(4) Again, let $\mathbb{T}$ be an arbitrary time scale. We let $f(t)=t$. We claim that $f^{\Delta}(t)=1$. To see this, first let $\varepsilon>0$ and note that $f(\sigma(t))=\sigma(t)$. So

$$|f(\sigma(t))-f(s)-1\cdot(\sigma(t)-s)|=|\sigma(t)-s-(\sigma(t)-s)|=0\leqslant\varepsilon|\sigma(t)-s|$$

which holds *for all* $s\in\mathbb{T}$.

The derivative rules given in the following theorem are proven in Bohner and Peterson [2].

Theorem 9. *Assume* $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^\kappa$. *Then*:

(i) *The sum* $f + g : \mathbb{T} \to \mathbb{R}$ *is differentiable at t with*

$$(f+g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) *For any constant* $\alpha \in \mathbb{R}$, $\alpha f : \mathbb{T} \to \mathbb{R}$ *is differentiable at t with*

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(iii) *The product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at t with*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

(iv) *If* $f(t)f(\sigma(t)) \neq 0$, *then* $\frac{1}{f}$ *is differentiable at t with*

$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}.$$

(v) *If* $g(t)g(\sigma(t)) \neq 0$, *then* $\frac{f}{g}$ *is differentiable at t with*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

The final thing that we use is the Δ-integral. It is defined as follows:

Definition 7. Let *f* be a bounded function on $[a, b)$ and let $P \in \mathscr{P}(a, b)$ be given by $a = t_0 < t_1 < \cdots < t_n = b$. In each interval $[t_{i-1}, t_i)$ with $1 \leqslant i \leqslant n$, choose an arbitrary point ξ_i and form the sum

$$S = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

S is a Riemann Δ-sum of *f* with partition $P \in \mathscr{P}$. *f* is *Riemann integrable* on $[a, b]$ if there exists a number *I* such that for all $\varepsilon > 0$, there exists $\delta > 0$ so that

$$|S - I| < \varepsilon$$

for every Riemann Δ-sum *S* of *f* corresponding to any $P \in \mathscr{P}_\delta(a, b)$ and independent of the choice of $\xi_i \in [t_{i-1}, t_i)$ for $1 \leqslant i \leqslant n$. The number *I* is called the *Riemann* Δ*-integral* of *f* from *a* to *b*.

It is well known that the standard Fundamental Theorem of Calculus holds for the time scale case, i.e. the Δ-integral conforms to the notion of evaluating antiderivatives at endpoints. It is also well known that the definition of the Riemann Δ-integral corresponds to the corresponding usual Riemann integral in the continuous case. Finally, measure theory allows the extension of the integral to that of the Lebesgue integral in which measurable sets and measurable functions are considered.

The integration by parts formula is now offered:

Theorem 10 (*Integration by Parts*). *Let u and v be continuous functions on* $[a, b]$ *that are* Δ*-differentiable on* $[a, b)$. *If* u^Δ *and* v^Δ *are integrable from a to b, then*

$$\int_a^b u^\Delta(t)v(t)\Delta t + \int_a^b u^\sigma(t)v^\Delta(t)\Delta t = u(b)v(b) - u(a)v(a).$$

Example 8.

(1) Let $\mathbb{T} = h\mathbb{Z}$. We wish to compute $\int_a^b t\Delta t$. Now,

$$\int_a^b t\Delta t = \frac{1}{2}\int_a^b 2t\Delta t = \frac{1}{2}\int_a^b (2t+h) - h\Delta t = \frac{1}{2}\left[\int_a^b (2t+h)\Delta t - \int_a^b h\Delta t\right].$$

Then, since $F(t) = t^2$ is differentiable with derivative $F^\Delta(t) = 2t + h$ for all $a, b \in \mathbb{T}$ by Example 7 part 1, the Fundamental Theorem of Calculus implies that

$$\int_a^b (2t+h)\Delta t = t^2\Big|_a^b = b^2 - a^2$$

holds for all $a, b \in \mathbb{T}$. The second integral in the difference above is now found:

$$\int_a^b h\Delta t = h(b-a).$$

From this, we deduce that

$$\int_a^b t\Delta t = \frac{(b^2 - a^2) - (h(b-a))}{2}$$

which holds for all $a, b \in \mathbb{T}$.

(2) Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$. We wish to compute $\int_a^b t^2\Delta t$. Note that as q is a constant, we have that

$$\int_a^b t^2\Delta t = \frac{1}{q^2+q+1}\int_a^b t^2(q^2+q+1)\Delta t.$$

Then, since $F(t) = t^3$ is differentiable with derivative $F^\Delta(t) = t^2(q^2+q+1)$ for all $a, b \in \mathbb{T}$ by Example 7 part 2, the Fundamental Theorem of Calculus implies that

$$\frac{1}{q^2+q+1}\int_a^b t^2(q^2+q+1)\Delta t = \frac{t^3}{q^2+q+1}\Bigg|_a^b = \frac{b^3-a^3}{q^2+q+1}$$

and thus

$$\int_a^b t^2\Delta t = \frac{b^3-a^3}{q^2+q+1}$$

for all $a, b \in \mathbb{T}$.

With derivatives and integrals defined, we can now offer one final definition: that of the generalized polynomials.

Definition 8. Define the *generalized polynomials* recursively as follows:

$$h_0(t, s) \equiv 1 \text{ for all } s, t \in \mathbb{T}$$

and

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau \quad \text{for all } s, t \in \mathbb{T}.$$

According to this definition, it follows that by letting $h_k^{\Delta}(t, s)$ denote the derivative of $h_k(t, s)$ with respect to t for fixed s then

$$h_k^{\Delta}(t, s) = h_{k-1}(t, s) \text{ for } k \in \mathbb{N}, t \in \mathbb{T}^{\kappa}.$$

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