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Instability results for slowly time varying linear dynamic systems on time scales

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Abstract

We develop eigenvalue criteria under which the solutions of a "slowly" time varying linear dynamic system of the form $x^{\Delta}(t) = A(t)x(t)$ are unstable.

Keywords: Stability; Time scale; Lyapunov; Time varying; Linear system; Slowly varying; Continuous; Discrete

1. Introduction

It is widely known that the stability characteristics of an autonomous linear system of differential or difference equations can be characterized completely by the placement of the eigenvalues of the system matrix [12]. Recently, Pötzsche, Siegmund, and Wirth [18] authored a landmark paper which developed necessary and sufficient conditions for the stability of time invariant linear systems on arbitrary time scales. Their characterization included the sufficient condition that the eigenvalues of the system matrix be contained in the possibly disconnected set of stability $\mathcal{S}(\mathbb{T}) \subset \mathbb{C}^-$, which may change for each time scale on which the system is studied. In [7,9], sufficient conditions are given on the placement of the eigenvalues of a sufficiently slowly time varying system which ensures exponential stability of the system solution.

The intent of this paper is to extend the classic results of the instability criteria for eigenvalue placement of a sufficiently slowly time varying system to the more general case of nonautonomous linear dynamic systems on a large class of time scales (i.e. those time scales with

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bounded graininess and $\sup \mathbb{T} = \infty$). For a brief introduction to time scales analysis, as well as necessary definitions for this paper, see [1,3,4]. We show by example that the placement of eigenvalues of the system matrix outside of the corresponding Hilger circle does not guarantee the instability of the time varying system, as is the case with autonomous linear systems of differential and difference equations [5,12–15,20]. We unify and extend the theorems of eigenvalue placement in the proper region of the complex plane for sufficiently slowly varying system matrices of continuous and discrete nonautonomous systems, which guarantees instability of the system, as in the classic papers of Wu [22] and Skoog and Lau [16]. To develop this theory for nonautonomous systems, we implement the generalized time scales version of the "second (direct) method" of A.M. Lyapunov [17] which yields an instability criterion result, as in the standard papers on stability of continuous and discrete dynamical systems by Kalman and Bertram [13,14] as well as the very recent paper [9]. The inherent beauty and elegance of Lyapunov's "second method" is that knowledge of the exact solution is not necessary. The qualitative behavior of the solution to the system (i.e. the stability or instability) can be investigated without computing the actual solution.

This paper is organized as follows. In Section 2, we give general definitions of our matrix norms, as well as stability definitions and characterizations. Section 3 introduces the unified time scale Lyapunov function for use in determining uniform exponential stability of linear systems on time scales. It also introduces a theorem that gives conditions on the eigenvalues of a sufficiently "slowly" time varying system matrix which ensures exponential stability of the system solution. Lastly, in this section we demonstrate how the quadratic Lyapunov function can also be used to determine the instability of a system. In Section 4, we give two theorems which characterize the instability of sufficiently slowly varying system by observing the placement of the system matrix eigenvalues in the Hilger complex plane. In the conclusions, we summarize our unified results.

2. General definitions

We start by introducing definitions and notation that will be employed in the sequel.

The *Euclidean norm* of an $n \times 1$ vector $x(t)$ is defined to be a real-valued function of t and is denoted by

$$\|x(t)\| = \sqrt{x^T(t)x(t)}.$$

The *induced norm* of an $m \times n$ matrix A is defined to be

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

The norm of A induced by the Euclidean norm above is equal to the nonnegative square root of the absolute value of the largest eigenvalue of the symmetric matrix $A^T A$. Thus, we define this norm next. The *spectral norm* of an $m \times n$ matrix A is defined to be

$$\|A\| = \left[\max_{\|x\|=1} x^T A^T A x\right]^{1/2}.$$

This will be the matrix norm that is used in the sequel and will be denoted by $\|\cdot\|$.

A symmetric matrix M is defined to be *positive semidefinite* if for all $n \times 1$ vectors x we have $x^T M x \geqslant 0$ and it is *positive definite* if $x^T M x \geqslant 0$, with equality only when $x = 0$. Negative semidefiniteness and definiteness are defined in terms of positive definiteness of $-M$.

We now define the concepts of uniform stability and uniform exponential stability. These two concepts involve the boundedness of the solutions of the regressive time varying linear dynamic equation

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0, \quad t_0 \in \mathbb{T}. \tag{2.1}$$

Definition 2.1. The time varying linear dynamic equation (2.1) is *uniformly stable* if there exists a finite constant $\gamma > 0$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$\|x(t)\| \leqslant \gamma \|x(t_0)\|, \quad t \geqslant t_0. \tag{2.2}$$

For the next definition, we define a stability property that not only concerns the boundedness of a solutions to (2.1), but also the asymptotic characteristics of the solutions as well. If the solutions to (2.1) possess the following stability property, then the solutions approach zero exponentially as $t \to \infty$ (i.e. the norms of the solutions are bounded above by a decaying exponential function).

Definition 2.2. The time varying linear dynamic equation (2.1) is called *uniformly exponentially stable* if there exist constants $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$\|x(t)\| \leqslant \|x(t_0)\| \gamma e_{-\lambda}(t, t_0), \quad t \geqslant t_0. \tag{2.3}$$

It is obvious by inspection of the previous definitions that we must have $\gamma \geqslant 1$. By using the word uniform, it is implied that the choice of γ does not depend on the initial time t_0.

The last stability definition given uses a uniformity condition to conclude exponential stability.

Definition 2.3. The linear state equation (2.1) is defined to be *uniformly asymptotically stable* if it is uniformly stable and given any $\delta > 0$, there exists a $T > 0$ so that for any t_0 and $x(t_0)$, the corresponding solution $x(t)$ satisfies

$$\|x(t)\| \leqslant \delta \|x(t_0)\|, \quad t \geqslant t_0 + T. \tag{2.4}$$

It is noted that the time T that must pass before the norm of the solution satisfies (2.4) and the constant $\delta > 0$ is independent of the initial time t_0.

We now state three theorems, in which the first two characterize uniform stability and uniform exponential stability in terms of the transition matrix for the system (2.1). Detailed explanations and proofs of the following theorems can be found in [8,9].

Theorem 2.1. *The time varying linear dynamic equation* (2.1) *is uniformly stable if and only if there exists a* $\gamma > 0$ *such that*

$$\|\Phi_A(t, t_0)\| \leqslant \gamma$$

for all $t \geqslant t_0$ *with* $t, t_0 \in \mathbb{T}$.

Theorem 2.2. *The time varying linear dynamic equation* (2.1) *is uniformly exponentially stable if and only if there exist* $\lambda, \gamma > 0$ *with* $-\lambda \in \mathcal{R}^+$ *such that*

$$\|\Phi_A(t, t_0)\| \leqslant \gamma e_{-\lambda}(t, t_0)$$

for all $t \geqslant t_0$ *with* $t, t_0 \in \mathbb{T}$.

Theorem 2.3. *The linear state equation* (2.1) *is uniformly exponentially stable if and only if it is uniformly asymptotically stable.*

3. Stability of slowly time varying linear dynamic systems and an instability criterion

In this section, we investigate the instability of the regressive "slowly" time varying linear dynamic system of the form

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0, \quad t_0 \in \mathbb{T}. \tag{3.1}$$

Our goal is to assess the instability of the unforced system by observing the system's total energy as the state of the system evolves in time. We assume that the time scale $\mathbb{T}$ is unbounded above.

To formalize our discussion, we employ time-dependent quadratic forms that are useful for analyzing stability. We will refer to these quadratic forms as *unified time scale quadratic Lyapunov functions*. For a symmetric matrix $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ we write the general quadratic Lyapunov function as $x^T(t)Q(t)x(t)$. If $x(t)$ is a solution to (3.1), and since $x^T(t)Q(t)x(t)$ has a scalar output, our interest lies in the behavior of the quantity $x^T(t)Q(t)x(t)$ for $t \geqslant t_0$. With this we now define one of the main ideas of this paper.

Definition 3.1. Let $Q(t)$ be a symmetric matrix such that $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$. A *unified time scale quadratic Lyapunov function* is given by

$$x^T(t)Q(t)x(t), \quad t \geqslant t_0, \tag{3.2}$$

with delta derivative

$$\begin{aligned}
&\big[x^T(t)Q(t)x(t)\big]^{\Delta_t} \\
&\quad = x^T(t)\big[A^T(t)Q(t) + \big(I + \mu(t)A^T(t)\big)\big(Q^{\Delta}(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t)\big)\big]x(t) \\
&\quad = x^T(t)\big[A^T(t)Q(t) + Q(t)A(t) + \mu(t)A^T(t)Q(t)A(t) \\
&\qquad + \big(I + \mu(t)A^T(t)\big)Q^{\Delta}(t)\big(I + \mu(t)A(t)\big)\big]x(t).
\end{aligned}$$

The matrix dynamic equation that is obtained by differentiating (3.2) with respect to t is given by

$$\begin{aligned}
&A^T(t)Q(t) + Q(t)A(t) + \mu(t)A^T(t)Q(t)A(t) \\
&\quad + \big(I + \mu(t)A^T(t)\big)Q^{\Delta}(t)\big(I + \mu(t)A(t)\big) = -M, \quad M = M^T.
\end{aligned}$$

One can easily see that it merges with the familiar continuous matrix differential equation ($\mathbb{T} = \mathbb{R}$) and discrete ($\mathbb{T} = \mathbb{Z}$) difference (recursive) equation obtained from the respective quadratic Lyapunov functions in $\mathbb{R}$ and $\mathbb{Z}$.

The unified time scale matrix dynamic equation merges into the continuous and discrete cases easily because of the time varying graininess $\mu(t)$. This unified time scale matrix dynamic equation not only unifies the two special cases of continuous and discrete time, it also extends these notions for arbitrary time scales $\mathbb{T}$, and as such plays a crucial role in our analysis.

First, we give a closed form for the unique, symmetric, and positive definite solution matrix to the *time scale Lyapunov matrix equation*

$$A^T(t)Q(t) + Q(t)A(t) + \mu(t)A^T(t)Q(t)A(t) = -M. \tag{3.3}$$

Remark. We note that the time scale Lyapunov matrix equation is the unification (with $B(t) \equiv A^T(t)$) of the Sylvester matrix equation [2]

$$XA(t) + B(t)X = -M$$

for the case $\mathbb{T} = \mathbb{R}$ and the Stein equation

$$B(t)XA(t) - X = -M$$

for the case $\mathbb{T} = \mathbb{Z}$. The Stein matrix equation above is written assuming that one is using recursive form. It can easily be transformed into an equivalent difference form

$$XA(t) + B(t)X + B(t)XA(t) = -M.$$

To prove that the matrix $Q(t)$ is a solution to the time scale Lyapunov matrix equation (3.3), we first state the following lemma that can be found in [4].

Lemma 3.1. *Suppose $A \in \mathcal{R}$ and C is a constant matrix. If C commutes with $A(t)$, then C commutes with $e_{A(t)}$. In particular, if $A(t)$ is constant matrix with respect to $e_{A(t)}$, then $A(t)$ commutes with $e_{A(t)}$.*

Now we present one of the main results of [9].

Theorem 3.1. *If the $n \times n$ matrix $A(t)$ has all eigenvalues in the corresponding Hilger circle for every $t \geqslant t_0$, then for each $t \in \mathbb{T}$, there exists some time scale $\mathbb{S}$ such that integration over $I := [0, \infty)_{\mathbb{S}}$ yields a unique solution to* (3.3) *given by*

$$Q(t) = \int_I e_{A^T(t)}(s, 0) M e_{A(t)}(s, 0) \Delta s. \tag{3.4}$$

Moreover, if M is positive definite, then $Q(t)$ is positive definite for all $t \geqslant t_0$.

The placement of eigenvalues in the complex plane of a time invariant matrix is a necessary and sufficient condition to ensure the stability and/or exponential stability of the system. This is a well-known fact in the theory of differential equations and difference equations, and it is investigated in depth in the landmark paper on the stability of time invariant linear systems on time scales by Pötzsche, Siegmund, and Wirth [18].

However, eigenvalue placement alone is neither necessary nor sufficient in the general case of any time varying linear dynamic system. Texts such as Brogan [5], Chen [6], and Rugh [20] give examples of time varying systems with "frozen" (time invariant) eigenvalues with negative real parts as well as bounded system matrices that still exhibit instability. The classic papers by Desoer [10], Rosenbrock [19], and a recent paper by Solo [21] demonstrate this fact for systems of differential equations, but they do show that under certain conditions, such as a bounded and sufficiently slowly varying system matrix, exponential stability can be obtained with correct eigenvalue placement in the complex plane. Desoer also published a similar paper [11] (a discrete analog to [10]) which illustrates the same instability characteristic of time varying systems in the discrete setting, but remedies the situation in essentially the same manner, with a bounded and sufficiently slowly varying system matrix.

To begin, we state a definition from Pötzsche, Siegmund, and Wirth's paper [18], in which the stability region for time invariant linear systems on time scales is introduced. This definition essentially says if the time average of the constant $\lambda \in \mathbb{C}$ is negative and $1 + \mu(t)\lambda \neq 0$

for all $t \in \mathbb{T}^\kappa$, then λ resides in the *regressive set of exponential stability* $\mathcal{S}(\mathbb{T})$, defined below. This definition is an integral part of the requirement for exponential stability of a time invariant linear system on an arbitrary time scale. If, for all $i = 1, \ldots, n$, $\lambda_i \in \mathcal{S}(\mathbb{T})$ and are *uniformly regressive* (there exists a positive constant δ such that $0 < \delta^{-1} \leqslant |1 + \mu(t)\lambda_i|$, $t \in \mathbb{T}^\kappa$), then the system (2.1), with $A(t) \equiv A$ constant, is uniformly exponentially stable (i.e. there exists an $\alpha > 0$ such that for any $t_0 \in \mathbb{T}$, $\gamma > 0$ can be chosen independently of t_0 such that $\|\Phi_A(t, t_0)\| \leqslant \|x(t_0)\|\gamma e^{-\alpha(t-t_0)}$).

Definition 3.2. [18] The *regressive set of exponential stability* for the dynamic system (2.1) when $A(t) \equiv A$ is a constant, is defined to be the set

$$\mathcal{S}(\mathbb{T}) = \left\{ \lambda \in \mathbb{C}\colon \limsup_{T\to\infty} \frac{1}{T - t_0} \int_{t_0}^{T} \lim_{s \searrow \mu(\tau)} \frac{\log|1 + s\lambda|}{s} \Delta\tau < 0 \right\}.$$

The regressive set of exponential stability is contained in $\{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) < 0\}$ at all times. The reader is referred to [18] for more explanation.

In the main theorem that follows, we require the eigenvalues $\lambda_i(t)$ of the time varying matrix $A(t)$ to reside in the corresponding Hilger circle for all $t \geqslant t_0$ and $i = 1, \ldots, n$. We note that the Hilger circle is defined as the set

$$\left\{ \lambda \in \mathbb{C}\colon \left| \frac{1}{\mu(t)} + \lambda(t) \right| < \frac{1}{\mu(t)} \right\} \subset \mathcal{S}(\mathbb{T}).$$

We now present the theorem for uniform exponential stability of slowly time varying systems which involves an eigenvalue condition on the time varying matrix $A(t)$ as well as the requirement that $A(t)$ is norm bounded and varies at a sufficiently slow rate (i.e. $\|A^\Delta(t)\| \leqslant \beta$, for some positive constant β and all $t \in \mathbb{T}$). This theorem is one of the main results of [9].

Theorem 3.2 *(Exponential stability for slowly time varying systems). Suppose for the regressive time varying linear dynamic system* (3.1) *with* $A(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *we have* $\mu_{\max}, \mu^\Delta_{\max} < \infty$, *there exists a constant* $\alpha > 0$ *such that* $\|A(t)\| \leqslant \alpha$, *and there exists a constant* $0 < \varepsilon < \frac{1}{\mu_{\max}} \leqslant \frac{1}{\mu(t)}$ *such that for every pointwise eigenvalue* $\lambda_i(t)$ *of* $A(t)$, $\operatorname{Re}_\mu[\lambda_i(t)] \leqslant -\varepsilon < 0$. *Then there exists a* $\beta > 0$ *such that if* $\|A^\Delta(t)\| \leqslant \beta$, (3.1) *is uniformly exponentially stable.*

We can also employ the unified timescale quadratic Lyapunov function to determine if the system (3.1) is unstable. This is a very useful result in the case where the development of a suitable matrix $Q(t)$ is difficult and the possibility of an unstable system begins to arise. One type of instability criteria is developed in the next theorem.

Theorem 3.3. [9] *Suppose there exists an* $n \times n$ *matrix* $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}$ *that is symmetric for all* $t \in \mathbb{T}$ *and has the following two properties*:

(i) $\|Q(t)\| \leqslant \rho$,
(ii) $A^T(t)Q(t) + (I + \mu(t)A^T(t))(Q^\Delta(t) + Q(t)A(t) + \mu(t)Q^\Delta(t)A(t)) \leqslant -\nu I$,

where $\rho, \nu > 0$. *Also suppose that there exists some* $t_a \in \mathbb{T}$ *such that* $Q(t_a)$ *is not positive semidefinite. Then the linear dynamic equation* (3.1) *is not uniformly stable.*

4. Instability of slowly time varying systems

We now consider time scales with such that $\sup \mathbb{T} = \infty$, $\inf \mathbb{T} = -\infty$ and $\mu(t)$, $\mu^\Delta(t)$ exist and are bounded above by the positive constants $\mu_{\max}$ and $\mu^\Delta_{\max}$, respectively. In the following theorem, we state a decomposition of $\mathbb{R}^n$ into two time varying subspaces that are invariant under the matrix $A(t)$.

Theorem 4.1. *Let the matrix-valued function $A(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ and satisfy the following conditions.*

(1) $\alpha = \sup_{t\geqslant t_0} \|A(t)\| < \infty$ *and* $\alpha^\Delta = \sup_{t\geqslant t_0} \|A^\Delta(t)\| < \infty$.
(2) *The eigenvalues of $A(t)$, $\lambda_1, \dots, \lambda_n$, are bounded away from a closed Jordan curve Γ in the complex plane for all $t \geqslant t_0$. The set $\Lambda_1(t) = \{\lambda_1(t), \dots, \lambda_k(t)\}$ lies inside Γ and the set $\Lambda_1(t) = \{\lambda_{k+1}(t), \dots, \lambda_n(t)\}$ lies outside Γ.*

Then there exists a matrix valued function $T(t) \in \mathrm{C}^1_{\mathrm{rd}}$ such that

(1)
$$\rho = \sup_{t\geqslant t_0} \|T(t)\| < \infty,$$
$$\tilde{\rho} = \sup_{t\geqslant t_0} \|T^{-1}(t)\| < \infty,$$
$$\rho^\Delta = \sup_{t\geqslant t_0} \|T^\Delta(t)\| \leqslant K\alpha^\Delta,$$

where ρ, $\tilde{\rho}$, ρ^Δ, α and K are finite positive constants and

(2) $T^{-1}(t)A(t)T(t) = \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix}$,

where $A_1(t)$ is $k \times k$ and its eigenvalues are $\lambda_1(t), \dots, \lambda_k(t)$ and $A_2(t)$ is $(n-k)\times(n-k)$ and its eigenvalues are $\lambda_{k+1}(t), \dots, \lambda_n(t)$. Also, $\sup_{t\geqslant t_0} \|A_1^\Delta(t)\| \leqslant K_1\alpha^\Delta$, and $\sup_{t\geqslant t_0} \|A_2^\Delta(t)\| \leqslant K_2\alpha^\Delta$, for some $0 < K_1, K_2 < \infty$.

Theorem 4.2. *Let $A(t)$ satisfy the conditions of Theorem 4.1, and suppose that $\mu_{\max}, \mu^\Delta_{\max} < \infty$, the eigenvalues $\lambda_1(t), \dots, \lambda_k(t)$ lie in $\mathrm{Re}_\mu[\lambda_i(t)] < -\varepsilon_1 < 0$ with $0 < \varepsilon_1 < \frac{1}{\mu_{\max}} < \frac{1}{\mu(t)}$ for all $t \geqslant t_0$, and that $\lambda_{k+1}(t), \dots, \lambda_n(t)$ lie in $\mathrm{Re}_\mu[\lambda_i(t)] > \varepsilon_2 > 0$ for all $t \geqslant t_0$. Then if α^Δ is sufficiently small, the zero solution of (3.1) is unstable.*

Proof. From Theorem 4.1 it is known that there exists a matrix valued function $T(t)$ such that conclusion (2) of the theorem holds. Let the matrices $Q_1(t)$ and $Q_2(t)$ be the respective solutions of

$$A_1^T(t)Q_1(t) + Q_1(t)A_1(t) + \mu(t)A_1^T(t)Q_1(t)A_1(t) = -I_{k\times k} \tag{4.1}$$

and

$$A_2^T(t)Q_2(t) + Q_2(t)A_2(t) + \mu(t)A_2^T(t)Q_2(t)A_2(t) = -I_{(n-k)\times(n-k)}. \tag{4.2}$$

Since the eigenvalues of $A_1(t)$ are in $\mathrm{Re}_\mu[\lambda_i(t)] < -\varepsilon_1 < 0$ for all $t \geqslant t_0$ and the eigenvalues of $A_2(t)$ are in $\mathrm{Re}_\mu[\lambda_i(t)] > \varepsilon_2 > 0$ for all $t \geqslant t_0$, it is known that (4.1) and (4.2) have unique solutions for all $t \in \mathbb{T}$. It is shown in [9] that the respective unique solutions are given by

$$Q_1(t) = \int_0^\infty e_{A_1^T(t)}(s,0)e_{A_1(t)}(s,0)\Delta s \tag{4.3}$$

and

$$Q_2(t) = -\int_{-\infty}^0 e_{A_2^T(t)}(s,0)e_{A_2(t)}(s,0)\Delta s, \tag{4.4}$$

with integration over $\mathbb{S} := \mu(t)\cdot\mathbb{Z}$ in both integrals. By construction, $Q_1(t)$ is positive definite and $Q_2(t)$ is negative definite for all $t \geqslant t_0$.

Applying the change of variables $z(t) = T^{-1}(t)x(t)$, we obtain

$$z^\Delta(t) = \bigl[T^{\sigma^{-1}}(t)A(t)T(t) - T^{\sigma^{-1}}(t)T^\Delta(t)\bigr]z(t), \qquad z(t_0) = z_0 = T^{-1}(t_0)x_0. \tag{4.5}$$

For the system in (4.5), we choose the unified quadratic Lyapunov function $V(t) := z^T(t)Q(t)z(t)$ where

$$Q(t) = \begin{bmatrix} Q_1(t) & 0 \\ 0 & Q_2(t) \end{bmatrix}. \tag{4.6}$$

Setting $\tilde{A}(t) := [T^{\sigma^{-1}}(t)A(t)T(t) - T^{\sigma^{-1}}(t)T^\Delta(t)]$, with t-dependence omitted for brevity, we see that along the solutions of (4.5),

$$\begin{aligned} V^\Delta &= z^T\bigl[\tilde{A}^TQ + Q\tilde{A} + \mu\tilde{A}^TQ\tilde{A} + \bigl(I + \mu\tilde{A}^T\bigr)Q^\Delta(I + \mu\tilde{A})\bigr]z \\ &= -z^TIz + z^T[M + F]z, \end{aligned} \tag{4.7}$$

where the first term $-z^TIz$ results from $T^\sigma(t)$ acting on $A(t)$ as the transformation matrix $T(t)$ in Theorem 4.1, $M(t) := (I + \mu(t)\tilde{A}^T(t))Q^\Delta(t)(I + \mu(t)\tilde{A}(t))$, the definition of $Q(t)$ is from Eq. (4.6), and the matrix $F(t)$, again with t-dependence omitted for brevity, is defined by

$$\begin{aligned} F := \mu\bigl[&-\bigl(T^{\sigma^{-1}}AT^\sigma\bigr)^TQ\bigl(\mu T^\sigma AT^\Delta + T^{\sigma^{-1}}T\bigr) - \bigl(\mu T^\sigma AT^\Delta + T^{\sigma^{-1}}T\bigr)^TQ\bigl(T^{\sigma^{-1}}AT^\sigma\bigr) \\ &+ \bigl(\mu T^\sigma AT^\Delta + T^{\sigma^{-1}}T\bigr)^TQ\bigl(\mu T^\sigma AT^\Delta + T^{\sigma^{-1}}T\bigr)\bigr] \\ &- Q\bigl(\mu T^\sigma AT^\Delta + T^{\sigma^{-1}}T\bigr) - \bigl(\mu T^\sigma AT^\Delta + T^{\sigma^{-1}}T\bigr)^TQ. \end{aligned}$$

Further, the matrix valued function $F(t)$ is bounded in norm above by some positive constant D, i.e. $\sup_{t\geqslant t_0}\|F(t)\| \leqslant D < \infty$, $\lim_{\mu\to 0}[F(t) + M(t)] = \dot{Q}(t) - Q(t)(T^{-1}(t)T(t)) - (T^{-1}(t)T(t))^TQ(t)$. From Theorem 4.1 we have $\|T^\Delta(t)\| \leqslant \alpha^\Delta$, $\|T^{-1}(t)\| \leqslant \tilde{\rho}$, and $\|Q(t)\|$ and $\|Q^\Delta(t)\|$ are bounded by construction in Eqs. (4.3) and (4.4) and [7], for all $t \geqslant t_0$. So there exists a constant $0 < C < \infty$ such that

$$\sup_{t\geqslant t_0}\bigl\|F(t) + M(t)\bigr\| \leqslant C\alpha^\Delta. \tag{4.8}$$

Thus, from (4.7) and (4.8), if $C\alpha^\Delta < 1$, then $V^\Delta(t) \leqslant -\nu z^T(t)z(t)$, for some $\nu > 0$. Since $Q_2(t)$ is negative definite for all $t \geqslant t_0$, the hypotheses of Theorem 3.3 are satisfied. Thus, if $\sup_{t\geqslant t_0}\|A^\Delta(t)\| = \alpha^\Delta$ is sufficiently small, the equilibrium solution of (4.5) is unstable which implies the equilibrium solution of (3.1) is unstable (by the stability preserving properties of a Lyapunov transformation of variables [8]). □

Another interesting result which follows easily from Theorem 4.2 has to do with nonlinear time varying perturbations on (3.1).

Theorem 4.3. *Consider the regressive nonlinear system*

$$x^{\Delta}(t) = A(t)x(t) + g\big(t, x(t)\big), \qquad x(t_0) = x_0, \tag{4.9}$$

where the matrix $A(t)$ satisfies the hypotheses of Theorem 4.2, $\mu_{\max}, \mu^{\Delta}_{\max} < \infty$, and the vector-valued function $g(t, x(t)) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfies $\|g(t, x(t))\| \leqslant \epsilon\|x(t)\|$ for all $t \in \mathbb{T}$ and $x(t)$. Then if $\sup_{t \geqslant t_0} \|A^{\Delta}(t)\|$ is sufficiently small, the zero solution of (4.9) is unstable.

For the final portion of this paper, we prove a new theorem that is a generalization of a result in [16], which encompasses the well-known result in differential equations as a special case of the time scale $\mathbb{T} = \mathbb{R}$.

Theorem 4.4. *Consider the system (3.1) and let $A(t)$ satisfy the assumptions of Theorem 4.2. Then for α^{Δ} sufficiently small there exists for each $t \geqslant t_0$ a k-dimensional subspace $S(t)$ such that, if $\phi(t)$ is a solution of (3.1) with $\phi(t_0) \in S(t_0)$, then $\|\phi(t)\| \leqslant \beta e_{-\gamma}(t, t_0)$ for some constants β, $\gamma > 0$ with $-\gamma \in \mathcal{R}^+$. Moreover, if $\phi(t_0) \notin S(t_0)$, then $\|\phi(t)\| \to \infty$ as $t \to \infty$.*

Proof. Let $\Phi_1(t, t_0)$ and $\Phi_2(t, t_0)$ be the transition matrices for $A_1(t)$ and $A_2(t)$, respectively. Since $\|A_1^{\Delta}(t)\| \leqslant K_1\alpha^{\Delta}$ and $A_1(t)$ has all eigenvalues such that $\mathrm{Re}_{\mu}[\lambda_i(t)] \leqslant -\varepsilon_1 < 0$ for all $i = 1, \ldots, k$ and $t \geqslant t_0$, by Theorem 3.2, if α^{Δ} is sufficiently small, then for some positive constants γ_1, c_1 with $-\gamma_1 \in \mathcal{R}^+$,

$$\big\|\Phi_1(t, t_0)\big\| \leqslant c_1 e_{-\gamma_1}(t, t_0), \quad t \geqslant t_0. \tag{4.10}$$

Also since $\|A_2^{\Delta}(t)\| \leqslant K_2\alpha^{\Delta}$ and $A_2(t)$ has all eigenvalues such that $\mathrm{Re}_{\mu}[\lambda_i(t)] \geqslant \varepsilon_2 > 0$ for all $i = k+1, \ldots, n$ and $t \geqslant t_0$, for some positive constants γ_2, c_2 we have,

$$\big\|\Phi_2(t, t_0)\big\| \leqslant c_2 e_{\gamma_2}(t, t_0), \quad t \leqslant t_0. \tag{4.11}$$

Define

$$\hat{\Phi}_1(t, t_0) = \begin{bmatrix} \Phi_1(t, t_0) & 0 \\ 0 & 0 \end{bmatrix} \tag{4.12}$$

and

$$\hat{\Phi}_2(t, t_0) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_2(t, t_0) \end{bmatrix}. \tag{4.13}$$

Now consider the system obtained after a Lyapunov transformation of variables $z(t) = T^{-1}(t)x(t)$ where the matrix $T^{\sigma}(t)$ acts on $A(t)$ as in Theorem 4.1. Then

$$z^{\Delta}(t) = \big[T^{\sigma^{-1}}(t)A(t)T^{\sigma}(t) - F(t)\big]z(t), \qquad z(t_0) = z_0 = T^{-1}(t_0)x_0, \tag{4.14}$$

where the function $F(t) = [\mu(t)T^{\sigma^{-1}}(t)A(t)T^{\Delta}(t) + T^{\sigma^{-1}}(t)T^{\Delta}(t)]$, and $\lim_{\mu \to 0} F(t) = T^{-1}(t)\dot{T}(t)$. We also have $\sup_{t \geqslant t_0} \|F(t)\| = \alpha^{\Delta}\tilde{\rho}K(1 + \alpha\mu_{\max})$. For each vector $y \in \mathbb{R}^n$, define the mapping J_y by

$$(J_y\phi)(t) := \hat{\Phi}_1(t,t_0)y - \int_{t_0}^{t} \hat{\Phi}_1\big(t,\sigma(\tau)\big)F(\tau)\phi(\tau)\Delta\tau + \int_{t}^{\infty} \hat{\Phi}_2\big(t,\sigma(\tau)\big)F(\tau)\phi(\tau)\Delta\tau. \tag{4.15}$$

From the bounds in (4.10) and (4.11), and the fact that $A(t)$, $T^{-1}(t)$, and $T^{\Delta}(t)$ are bounded for all $t \in \mathbb{T}$, it follows that J_y maps $L^n_\infty(t_0,\infty)_{\mathbb{T}}$ into itself (where $L^n_\infty(t_0,\infty)_{\mathbb{T}}$ is the normed linear space of $\mathbb{R}^n$-valued functions of t such that if $\psi \in L^n_\infty(t_0,\infty)_{\mathbb{T}}$, then $\sup_{t \geqslant t_0} \|\psi(t)\| < \infty$). We now show that for sufficiently small α^{Δ}, J_y is a contraction on $L^n_\infty(t_0,\infty)_{\mathbb{T}}$ and thus has a unique fixed point ϕ^*_y. This fixed point is a solution of (4.14).

To show that J_y is a contraction, observe

$$\begin{aligned}(J_y\phi_1)(t) - (J_y\phi_2)(t) = {}& -\int_{t_0}^{t} \hat{\Phi}_1\big(t,\sigma(\tau)\big)F(\tau)\big(\phi_1(\tau)-\phi_2(\tau)\big)\Delta\tau \\ &+ \int_{t}^{\infty} \hat{\Phi}_2\big(t,\sigma(\tau)\big)F(\tau)\big(\phi_1(\tau)-\phi_2(\tau)\big)\Delta\tau.\end{aligned} \tag{4.16}$$

Applying the norm $\|\cdot\|_\infty$ (the norm on $L^n_\infty(t_0,\infty)_{\mathbb{T}}$) to both sides of (4.16), we obtain

$$\begin{aligned}\|J_y\phi_1 - J_y\phi_2\|_\infty &\leqslant K\alpha^{\Delta}\tilde{\rho}(1+\alpha\mu_{\max})\|\phi_1-\phi_2\|_\infty \\ &\quad \times \left[c_1\int_{t_0}^{t} e_{-\gamma_1}\big(t,\sigma(\tau)\big)\Delta\tau + c_2\int_{t}^{\infty} e_{\gamma_2}\big(t,\sigma(\tau)\big)\Delta\tau\right] \\ &\leqslant \gamma K\alpha^{\Delta}\tilde{\rho}(1+\alpha\mu_{\max})\|\phi_1-\phi_2\|_\infty,\end{aligned} \tag{4.17}$$

where $\gamma := (\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2})$.

Thus, if $\gamma K\alpha^{\Delta}\tilde{\rho}(1+\alpha\mu_{\max}) < 1$, then J_y is a contraction and has unique fixed point ϕ^*_y. Using successive approximations, it can also be shown that

$$\big\|\phi^*_y(t)\big\| \leqslant \frac{1}{1-q}\big\|\hat{\Phi}_1(t,t_0)y\big\| \leqslant \frac{c_1\|y\|}{1-q}e_{-\gamma_1}(t,t_0), \tag{4.18}$$

where $q := \gamma K\alpha^{\Delta}\tilde{\rho}(1+\alpha\mu_{\max}) < 1$.

In the definition of J_y, only the first k components are involved. Then last $n-k$ components can be chosen to be zero. From (4.15) and the fact that ϕ^*_y is a fixed point, the first k components of $\phi^*_y(t_0)$ are the first k components of y and the remaining $n-k$ components are given by

$$\big[\phi^*_y(t_0)\big]_j = \left[\int_{t_0}^{\infty} \hat{\Phi}_2\big(t_0,\sigma(\tau)\big)F(\tau)\phi^*_y(\tau)\Delta\tau\right]_j, \quad j = k+1,\dots,n. \tag{4.19}$$

Thus, there exists a k parameter family of solutions of (4.14) that are exponentially bounded (in the sense of (4.18)). Since the system is linear, we have the following property that for $\phi^*_{y_1} + \phi^*_{y_2} = \phi^*_{y_1+y_2}$, and hence, for each t_0, there exists a k-dimensional subspace $\tilde{S}(t_0)$ of $\mathbb{R}^n$ such that if $\phi(t)$ is a solution of (4.14) with $\phi(t_0) \in \tilde{S}(t_0)$, then $\phi(t)$ is exponentially bounded. Transforming back to the system (3.1), the theorem follows with $S(t) := T(t)\tilde{S}(t)$.

It remains to show that if $\phi(t_0) \notin \tilde{S}(t_0)$ and ϕ is a solution of (4.14), then ϕ is unbounded. We suppose that $\|\phi(t)\|$ is bounded. By variation of constants,

$$\begin{aligned}\phi(t) &= \left[\hat{\Phi}_1(t,t_0) + \hat{\Phi}_2(t,t_0)\right]\phi(t_0) - \int_{t_0}^{t} \left[\hat{\Phi}_1\big(t,\sigma(\tau)\big) + \hat{\Phi}_2\big(t,\sigma(\tau)\big)\right] F(\tau)\phi(\tau)\,\Delta\tau \\ &= \hat{\Phi}_1(t,t_0)\phi(t_0) + \hat{\Phi}_2(t,t_0)\Omega - \int_{t_0}^{t} \hat{\Phi}_1\big(t,\sigma(\tau)\big)F(\tau)\phi(\tau)\,\Delta\tau \\ &\quad + \int_{t}^{\infty} \hat{\Phi}_2\big(t,\sigma(\tau)\big)F(\tau)\phi(\tau)\,\Delta\tau, \end{aligned} \tag{4.20}$$

where

$$\Omega = \phi(t_0) - \int_{t_0}^{\infty} \hat{\Phi}_2\big(t,\sigma(\tau)\big)F(\tau)\phi(\tau)\,\Delta\tau. \tag{4.21}$$

The integrals in (4.20) and (4.21) exist because of the assumption that $\|\phi(t)\|$ is bounded (by (4.10) and (4.11)). So if $\phi(t_0) \notin \tilde{S}(t_0)$, then the last $n-k$ components of Ω are not all zero. Thus $\hat{\Phi}_2(t,t_0)\Omega$ will be unbounded, which contradicts the fact assumption that ϕ is bounded. □

5. Conclusions

In this paper, we have unified and extended the results for an eigenvalue condition on sufficiently slowly varying linear dynamic systems on continuous domains to any closed subset of the real line, i.e. any time scale, to determine the instability of the system. If the eigenvalues of the system matrix are restricted from crossing the boundary of the (possibly dynamically changing) Hilger circle for all $t \geqslant t_0$, along with the system matrix varying at a sufficiently slow rate, then the stability (and in this paper, instability) characteristics of the time varying system can be analyzed using time invariant eigenvalue conditions.

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