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An exploration of combined dynamic derivatives on time scales and their applications

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Abstract

It is becoming evident that different dynamic derivatives play increasingly important roles in approximating functions and solutions of nonlinear differential equations for their great flexibility in grid designs. Different dynamic derivatives on time scales not only offer a convenient way in practical applications, but also show their distinctive features in approximations. It may be worthwhile to investigate if such useful features can be maintained or even improved in certain senses while different dynamic derivatives are used in the same application simultaneously. Under this consideration, we will introduce the combined delta (Δ, or forward) and nabla (∇, or backward) dynamic derivatives, explore their basic properties, and investigate their applications for approximating classical derivative functions and for solving differential equation problems in this paper. Proper forward jump, backward jump and step functions will be introduced and utilized. It is found that while the combined dynamic derivatives possess similar properties as Δ and ∇ derivatives, they offer more balanced approximations to the targeted functions and differential equations at satisfactory accuracy. The combined dynamic derivatives also reduce the unexpected computational spuriosity, and therefore lead to more reliable numerical algorithm designs. Computational examples are given to further illustrate our results.

MSC: 34B10; 39A10; 65D25

Keywords: Time scales; Dynamic derivatives and dynamic equations; Anti-derivative; Linear combination; Nonuniform grids; Boundary value problems; Approximation; Spuriosity

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1. Introduction

There has been a considerable amount of interest and recent publications in the theory and applications of dynamic derivatives on time scales. The study unifies traditional concepts of derivatives and differences. The investigations are not only significant in the theoretical research of differential and difference equations, but also crucial in certain computational applications such as adaptive computing and multiscale methods [3–8,16,17].

The primary purpose of this paper is to explore basic properties of the first and second order diamond-α ($\diamond_\alpha$) derivatives which are linear combinations of Δ and ∇ dynamic derivatives on time scales. A motivation of this study is for designing more balanced adaptive algorithms on nonuniform grids with reduced spuriosity. Similarities and differences between the diamond-α and standard dynamic derivatives will be investigated. We will extend the exploration to the approximation of classical derivative functions and solutions of dynamic equation boundary value problems via $\diamond_\alpha$ differentiation applications on time scales. The readers are referred to [2,3,10,11,15] for systematic studies of standard dynamic derivatives with calculations; to [3,4,8,10,11] for the latest developments in the theory of dynamic boundary value problems; and to [6,8,12–14] and references therein for recent discussions about numerical spuriosity.

It is known that patterns of solutions are fundamental in solving differential equation problems. To preserve such important patterns, nonuniform computational grids are essential and this leads to many well-known adaptive methods [8,16,17]. Since unwanted computational spuriosity is also introduced as a byproduct, a qualitative study of the fallacy becomes crucial [5–8]. To investigate this case, we may assume that all time scales considered in this paper are symmetric, since symmetry is one of the most important and basic geometric patterns considered. We will introduce the combined dynamic derivatives, and continue the study by obtaining proper chain rules and change of variable formulae for the integrals associated with diamond-α derivatives. As a consequence, we will show that while Δ and ∇ dynamic derivatives complement each other, the combined dynamic derivatives provide more balanced formulations for approximating classical derivatives. We will demonstrate that dynamic boundary value problems utilizing diamond-α derivatives offer better pattern preservations as compared with standard dynamic derivatives. Properly defined combined dynamic derivatives not only ensure the required accuracy, but also give numerical solutions with correct geometric patterns over the domain. The latter is particularly meaningful in computational applications.

Our discussions will be organized as follows. In Section 2, definitions of the diamond-α derivatives will be introduced. We will investigate basic properties of the combined dynamic derivatives, as well as differences between the combined and standard dynamic derivatives. Proper differentiation rules will be established for the $\diamond_\alpha$ derivatives. In the third section, we will define the corresponding combined, or $\diamond_\alpha$, integrals and then show the change of variable formulae for the integrals. We assume that the reader has some working experiences with the Δ and ∇ differentiations as well as integrals. In Section 4, we will carry out a number of computational experiments with the combined derivatives and integrals. Comparisons are given for the first and second order combined dynamic derivatives, and a two-point dynamic boundary value problem utilizing the $\diamond_\alpha$ derivatives will also be presented. The numerical experiments will further confirm our results and show the necessity and importance of combined dynamic derivative applications.

2. Combined dynamic derivatives

A one-dimensional time scale $\mathbb{T}$ is any closed subset of $\mathbb{R}$. The standard forward-jump and backward-jump functions σ, ρ can be defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively. The corresponding forward-step and backward-step functions are

$$\mu(t) = \sigma(t) - t, \quad \eta(t) = t - \rho(t),$$

respectively. For the sake of convenience, we set

$$a = \sup \mathbb{T}, \quad b = \inf \mathbb{T}.$$

A point $t \in \mathbb{T}$ is called left-scattered, right-scattered if $\rho(t) < t$, $\sigma(t) > t$, respectively. A point $t \in \mathbb{T}$ is called left-dense, right-dense if $\rho(t) = t$, $\sigma(t) = t$, respectively. We define $\mathbb{T}^{\kappa} = \mathbb{T}$ if b is left-dense and $\mathbb{T}^{\kappa} = \mathbb{T}\backslash\{b\}$ if b is left-scattered. Similarly, we define $\mathbb{T}_{\kappa} = \mathbb{T}$ if a is right-dense and $\mathbb{T}_{\kappa} = \mathbb{T}\backslash\{a\}$ if a is right-scattered. We denote $\mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa} = \mathbb{T}^{\kappa}_{\kappa}$. By the same token, we may in general define extended time scales $\mathbb{T}^{\kappa^m}$, $\mathbb{T}_{\kappa^n}$ and $\mathbb{T}^{\kappa^m}_{\kappa^n}$; $m, n = 0, 1, 2, \ldots,$ under the notation $\mathbb{T}^{\kappa^0} = \mathbb{T}_{\kappa^0} = \mathbb{T}$. We say that a function f defined on $\mathbb{T}$ is Δ differentiable on $\mathbb{T}^{\kappa}$ if for all $\varepsilon > 0$ there is a neighborhood U of $t \in \mathbb{T}^{\kappa}$ such that for some γ the inequality

$$|f(\sigma(t)) - f(s) - \gamma(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|$$

is true for all $s \in U$, and in this case we write $f^{\Delta}(t) = \gamma$. Similarly, we say that a function f defined on $\mathbb{T}$ is ∇ differentiable on $\mathbb{T}_{\kappa}$ if for $\varepsilon > 0$ there is a neighborhood V of t such that for some $\hat{\gamma}$ the inequality

$$|f(\rho(t)) - f(s) - \hat{\gamma}(\rho(t) - s)| < \varepsilon|\rho(t) - s|$$

is true for all $s \in V$, and in this case, we write $f^{\nabla}(t) = \hat{\gamma}$. To study higher order dynamic equations, we may define $\sigma^{m+1}(t) = \sigma^m(t)$, $\rho^{n+1}(t) = \rho^n(t)$, $f^{\Delta^{m+1}} = f^{\Delta^m}$, $f^{\nabla^{n+1}} = f^{\nabla^n}$ and $f^{\Delta^m\nabla^n} = (f^{\Delta^m})^{\nabla^n}$, $f^{\nabla^n\Delta^m} = (f^{\nabla^n})^{\Delta^m}$, where $m, n = 0, 1, 2, \ldots$ and $\sigma^0(t) = \rho^0(t) = t$, $f^{\Delta^0} = f^{\nabla^0} = f$. We may also write $f^{\sigma}(t) = f(\sigma(t))$, $f^{\rho}(t) = f(\rho(t))$. Throughout our discussion, we will assume that the time scale $\mathbb{T}$ used is bounded.

We refer the reader to [3,4] for a comprehensive development of the calculus of the Δ derivative and we refer the reader to [2] for an account of the calculus corresponding to the ∇ derivative. A number of useful dynamic derivative relations are obtained by Ahlbrandt et al. in the study of the change of variables formulae on time scales [1]. Based on existing results and recent investigations in [5–8], we may state the following.

Definition 2.1. Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}$ we define the diamond-α dynamic derivative $f^{\diamond_\alpha}(t)$ by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1 - \alpha) f^{\nabla}(t), \quad 0 \leqslant \alpha \leqslant 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

We may notice that the diamond-α derivative reduces to the standard Δ derivative as $\alpha=1$, or the standard ∇ derivative as $\alpha=0$, while it represents a "weighted dynamic derivative" for $\alpha \in (0,1)$. Furthermore, the combined dynamic derivative offers a centralized derivative formula on any uniformly discrete time scale $\mathbb{T}$ when $\alpha=\frac{1}{2}$. Needless to say, the latter feature is particularly useful in many computational applications.

Lemma 2.2 (*Chain Rule*). *Let $v:\mathbb{T}\to\mathbb{R}$ be monotone, $\tilde{\mathbb{T}}=v(\mathbb{T})$ be a time scale, $\tilde{\Delta}$ and $\tilde{\nabla}$ denote corresponding derivatives with respect to $\tilde{\mathbb{T}}$, and suppose $\omega:\tilde{\mathbb{T}}\to\mathbb{R}$. Let $\tilde{\Delta}$, $\tilde{\nabla}$ be defined on $\tilde{\mathbb{T}}$. Assume that $v^{\Delta}(t)$, v^{∇}, $\omega^{\tilde{\Delta}}(v(t))$ and $\omega^{\tilde{\nabla}}(v(t))$ exist.*

(i) *If v is strictly increasing then*

$$\begin{aligned}(\omega\circ v)^{\diamond_\alpha} &= ((\omega^{\tilde{\Delta}}\circ v)+(\omega^{\tilde{\nabla}}\circ v))v^{\diamond_\alpha}-\alpha(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}-(1-\alpha)(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}\\ &= (\omega^{\tilde{\diamond}_\alpha}\circ v)v^{\tilde{\diamond}_{1/2}}-\alpha(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}-(1-\alpha)(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}.\end{aligned}$$

(ii) *If v is strictly decreasing then*

$$\begin{aligned}(\omega\circ v)^{\diamond_\alpha} &= ((\omega^{\tilde{\nabla}}\circ v)+(\omega^{\tilde{\Delta}}\circ v))v^{\diamond_\alpha}-\alpha(\omega^{\tilde{\Delta}}\circ v)v^{\Delta}-(1-\alpha)(\omega^{\tilde{\nabla}}\circ v)v^{\nabla}\\ &= (\omega^{\tilde{\diamond}_{1-\alpha}}\circ v)v^{\tilde{\diamond}_{1/2}}-\alpha(\omega^{\tilde{\nabla}}\circ v)v^{\nabla}-(1-\alpha)(\omega^{\tilde{\Delta}}\circ v)v^{\Delta}.\end{aligned}$$

Proof. To see (i), it is observed that

$$\begin{aligned}(\omega\circ v)^{\diamond_\alpha} &= \alpha(\omega^{\tilde{\Delta}}\circ v)v^{\Delta}+(1-\alpha)(\omega^{\tilde{\nabla}}\circ v)v^{\nabla}\\ &= \alpha(\omega^{\tilde{\Delta}}\circ v)v^{\Delta}+(1-\alpha)(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}+(1-\alpha)(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}\\ &\quad +\alpha(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}-(1-\alpha)(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}-\alpha(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}\\ &= ((\omega^{\tilde{\Delta}}\circ v)+(\omega^{\tilde{\nabla}}\circ v))v^{\diamond_\alpha}-\alpha(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}-(1-\alpha)(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}\\ &= \alpha(\omega^{\tilde{\Delta}}\circ v)v^{\Delta}+(1-\alpha)(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}+(1-\alpha)(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}\\ &\quad +\alpha(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}-(1-\alpha)(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}-\alpha(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}\\ &= (\omega^{\tilde{\diamond}_\alpha}\circ v)v^{\Delta}+(\omega^{\tilde{\diamond}_\alpha}\circ v)v^{\nabla}-(1-\alpha)(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}-\alpha(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}\\ &= (\omega^{\tilde{\diamond}_\alpha}\circ v)(v^{\Delta}+v^{\nabla})-\alpha(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}-(1-\alpha)(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}\\ &= (\omega^{\tilde{\diamond}_\alpha}\circ v)v^{\tilde{\diamond}_{1/2}}-\alpha(\omega^{\tilde{\Delta}}\circ v)v^{\nabla}-(1-\alpha)(\omega^{\tilde{\nabla}}\circ v)v^{\Delta}.\end{aligned}$$

The proof of (ii) is similar. $\square$

Theorem 2.3. *Let $f,g:\mathbb{T}\to\mathbb{R}$ be diamond-α differentiable at $t\in\mathbb{T}$. Then*

(i) *$f+g:\mathbb{T}\to\mathbb{R}$ is diamond-α differentiable at $t\in\mathbb{T}$ with*

$$(f+g)^{\diamond_\alpha}(t)=f^{\diamond_\alpha}(t)+g^{\diamond_\alpha}(t).$$

(ii) *For any constant c, $cf:\mathbb{T}\to\mathbb{R}$ is diamond-α differentiable at $t\in\mathbb{T}$ with*

$$(cf)^{\diamond_\alpha}(t)=cf^{\diamond_\alpha}(t).$$

(iii) $fg : \mathbb{T} \to \mathbb{R}$ *is diamond-*α *differentiable at* $t \in \mathbb{T}$ *with*

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t).$$

(iv) *for* $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $1/g : \mathbb{T} \to \mathbb{R}$ *is diamond-*α *differentiable at* $t \in \mathbb{T}$ *with*

$$\left(\frac{1}{g}\right)^{\diamond_\alpha}(t) = -\frac{1}{g(t)g^{\sigma}(t)g^{\rho}(t)}((g^{\sigma}(t)+g^{\rho}(t))g^{\diamond_\alpha}(t) - \alpha g^{\Delta}(t)g^{\sigma}(t) - (1-\alpha)g^{\nabla}(t)g^{\rho}(t)).$$

(v) *for* $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $f/g : \mathbb{T} \to \mathbb{R}$ *is diamond-*α *differentiable at* $t \in \mathbb{T}$ *with*

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{1}{g(t)g^{\sigma}(t)g^{\rho}(t)}(f^{\diamond_\alpha}(t)g^{\sigma}(t)g^{\rho}(t) - \alpha f^{\sigma}(t)g^{\rho}(t)g^{\Delta}(t) - (1-\alpha)f^{\rho}(t)g^{\sigma}(t)g^{\nabla}(t)).$$

Proof. We only need to show (iii)–(v) since proofs of (i) and (ii) are straightforward from the definition of the diamond-α derivative. First,

$$\begin{aligned}(fg)^{\diamond_\alpha}(t) &= \alpha(fg)^{\Delta}(t) + (1-\alpha)(fg)^{\nabla}(t)\\ &= \alpha f^{\Delta}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\nabla}(t)g(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t)\\ &= f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t).\end{aligned}$$

Secondly, according to [3], we have

$$\begin{aligned}\left(\frac{1}{g}\right)^{\diamond_\alpha}(t) &= -\alpha\frac{g^{\Delta}(t)}{g(t)g^{\sigma}(t)} - (1-\alpha)\frac{g^{\nabla}(t)}{g(t)g^{\rho}(t)}\\ &= -\alpha\frac{g^{\Delta}(t)}{g(t)g^{\sigma}(t)} - (1-\alpha)\frac{g^{\nabla}(t)}{g(t)g^{\sigma}(t)} + (1-\alpha)\frac{g^{\nabla}(t)}{g(t)g^{\sigma}(t)}\\ &\quad - (1-\alpha)\frac{g^{\nabla}(t)}{g(t)g^{\rho}(t)} - \alpha\frac{g^{\Delta}(t)}{g(t)g^{\rho}(t)} + \alpha\frac{g^{\Delta}(t)}{g(t)g^{\rho}(t)}\\ &= -\frac{1}{g(t)g^{\sigma}(t)}(\alpha g^{\Delta}(t) + (1-\alpha)g^{\nabla}(t))\\ &\quad - \frac{1}{g(t)g^{\rho}(t)}(\alpha g^{\Delta}(t) + (1-\alpha)g^{\nabla}(t))\\ &\quad + \frac{1}{g(t)g^{\sigma}(t)g^{\rho}(t)}(\alpha g^{\Delta}(t)g^{\sigma}(t) + (1-\alpha)g^{\nabla}(t)g^{\rho}(t))\\ &= -\frac{1}{g(t)g^{\sigma}(t)g^{\rho}(t)}((g^{\sigma}(t)+g^{\rho}(t))g^{\diamond_\alpha}(t) - \alpha g^{\Delta}(t)g^{\sigma}(t)\\ &\quad - (1-\alpha)g^{\nabla}(t)g^{\rho}(t)).\end{aligned}$$

Finally, identity (v) follows as a consequence from (iii) and (iv) as $f(t)/g(t) = f(t) \cdot (1/g(t))$. $\square$

Theorem 2.4. *Let $f : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Then the following hold*:

(i) $f^{\diamond_\alpha \Delta}(t) = \alpha f^{\Delta\Delta}(t) + (1-\alpha) f^{\nabla\Delta}(t)$.
(ii) $f^{\diamond_\alpha \nabla}(t) = \alpha f^{\Delta\nabla}(t) + (1-\alpha) f^{\nabla\nabla}(t)$.
(iii) $f^{\Delta\diamond_\alpha}(t) = \alpha f^{\Delta\Delta}(t) + (1-\alpha) f^{\Delta\nabla}(t) \neq f^{\diamond_\alpha \Delta}(t)$.
(iv) $f^{\nabla\diamond_\alpha}(t) = \alpha f^{\nabla\Delta}(t) + (1-\alpha) f^{\nabla\nabla}(t) \neq f^{\diamond_\alpha \nabla}(t)$.
(v) $f^{\diamond_\alpha\diamond_\alpha}(t) = \alpha^2 f^{\Delta\Delta}(t) + \alpha(1-\alpha)(f^{\Delta\nabla}(t) + f^{\nabla\Delta}(t)) + (1-\alpha)^2 f^{\nabla\nabla}(t) \neq \alpha^2 f^{\Delta\Delta}(t) + (1-\alpha)^2 f^{\nabla\nabla}(t)$.

Proof. We only need to show (i), (iii) and (v) since proofs of the other identities are similar. Following the definition of the diamond-α differentiation, we have

$$\begin{aligned} f^{\diamond_\alpha \Delta}(t) &= (f^{\diamond_\alpha})^{\Delta}(t) = (\alpha f^{\Delta} + (1-\alpha)^{\nabla}(t))^{\Delta} \\ &= \alpha f^{\Delta\Delta}(t) + (1-\alpha) f^{\nabla\Delta}(t) \end{aligned}$$

and this shows (i). By the same token,

$$f^{\Delta\diamond_\alpha}(t) = \alpha f^{\Delta\Delta}(t) + (1-\alpha) f^{\Delta\nabla}(t) \neq f^{\diamond_\alpha\Delta}(t)$$

which ensures (iii). For (v), we observe that

$$\begin{aligned} f^{\diamond_\alpha\diamond_\alpha}(t) &= (\alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t))^{\diamond_\alpha} = \alpha(\alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t))^{\Delta} \\ &\quad + (1-\alpha)(\alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t))^{\nabla} \\ &= \alpha^2 f^{\Delta\Delta}(t) + \alpha(1-\alpha)(f^{\Delta\nabla}(t) + f^{\nabla\Delta}(t)) + (1-\alpha)^2 f^{\nabla\nabla}(t). \qquad \square \end{aligned}$$

3. Combined dynamic integrations

Definition 3.1 (*Anderson et al.* [2], *Bohner and Peterson* [3]). Let $a, t \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ antiderivative of f provided that $F^{\Delta}(t) = f(t)$ holds for $t \in \mathbb{T}$. We define the Δ integral of f by

$$\int_a^t f(\tau)\Delta\tau = F(t) - F(a), \quad t \in \mathbb{T}.$$

On the other hand, let $g : \mathbb{T} \to \mathbb{R}$. A function $G : \mathbb{T} \to \mathbb{R}$ is called a ∇ antiderivative of g provided that $G^{\nabla}(t) = g(t)$ holds for $t \in \mathbb{T}$. We define the ∇ integral of g by

$$\int_a^t g(\tau)\nabla\tau = G(t) - G(a), \quad t \in \mathbb{T}.$$

Definition 3.2. Let $a, t \in \mathbb{T}$, and $h : \mathbb{T} \to \mathbb{R}$. We define the $\diamondsuit_\alpha$ integral of h as

$$\int_a^t h(\tau)\diamondsuit_\alpha\tau = \alpha\int_a^t h(\tau)\Delta\tau + (1-\alpha)\int_a^t h(\tau)\nabla\tau, \quad t\in\mathbb{T},\ 0\leqslant\alpha\leqslant 1.$$

We may notice that since $\diamondsuit_\alpha$ integral is a combined Δ and ∇ integral, we in general do not have

$$\left(\int_a^t f(\tau)\diamondsuit_\alpha\tau\right)^{\diamondsuit_\alpha} = f(t), \quad t\in\mathbb{T}.$$

However, it may be interesting to see basic properties between different dynamic derivatives and the $\diamondsuit_\alpha$ integral. As it has been known, the interactions between different derivatives and integrals are closely tied to the function composition $\sigma(\rho(t))$ and $\rho(\sigma(t))$ [15]. Since $\sigma(\rho(t)) \neq t$ at points which are left-dense and right-scattered and $\rho(\sigma(t)) \neq t$ at points which are right-dense and left scattered, we need to consider these points separately in our investigations. For the convenience of discussion, we define the following sets:

$$\begin{aligned}
A &:= \{t\in\mathbb{T} : t \text{ is left-dense and right-scattered}\},\\
B &:= \{t\in\mathbb{T} : t \text{ is left-scattered and right-dense}\},\\
C &:= \{t\in\mathbb{T} : t \text{ is left-scattered and right-scattered}\},\\
D &:= \{t\in\mathbb{T} : t \text{ is left-dense and right-dense}\}.
\end{aligned}$$

The following lemma is a complete version of the similar result in [15].

Lemma 3.3. *Let $a, t \in \mathbb{T}$. If the left-sided limits of $f : \mathbb{T} \to \mathbb{R}$ exist at left-dense points in $\mathbb{T}$ then $\int_a^t f(\tau)\Delta\tau$ is ∇-differentiable on $\mathbb{T}$ and*

$$\left(\int_a^t f(\tau)\Delta\tau\right)^{\nabla} = \begin{cases} f(\rho(t)) & \text{if } t\in B\cup C,\\ \lim\limits_{\tau\to t^-} f(\tau) & \text{if } t\in A\cup D.\end{cases}$$

Further, if the right-sided limits of $g : \mathbb{T} \to \mathbb{R}$ exist at right-dense points in $\mathbb{T}$ then $\int_a^t g(\tau)\nabla\tau$ is Δ-differentiable on $\mathbb{T}$ and

$$\left(\int_a^t g(\tau)\nabla\tau\right)^{\Delta} = \begin{cases} g(\sigma(t)) & \text{if } t\in A\cup C,\\ \lim\limits_{\tau\to t^+} g(\tau) & \text{if } t\in B\cup D.\end{cases}$$

Lemma 3.4. *Let $a, t \in \mathbb{T}$. If the left-sided limits of $f : \mathbb{T} \to \mathbb{R}$ exist at left-dense points in $\mathbb{T}$ then $\int_a^t f(\tau)\Delta\tau$ is $\diamondsuit_\alpha$-differentiable on $\mathbb{T}$ and*

$$\left(\int_a^t f(\tau)\Delta\tau\right)^{\diamondsuit_\alpha} = \alpha f(t) + (1-\alpha)\begin{cases} f(\rho(t)) & \text{if } t\in B\cup C,\\ \lim\limits_{\tau\to t^-} f(\tau) & \text{if } t\in A\cup D.\end{cases}$$

Further, if the right-sided limits of $g : \mathbb{T} \to \mathbb{R}$ exist at right-dense points in $\mathbb{T}$ then $\int_a^t g(\tau)\nabla\tau$ is $\diamondsuit_\alpha$-differentiable on $\mathbb{T}$ and

$$\left(\int_a^t f(\tau)\nabla\tau\right)^{\diamondsuit_\alpha} = (1-\alpha) f(t) + \alpha\begin{cases} f(\sigma(t)) & \text{if } t\in A\cup C,\\ \lim\limits_{\tau\to t^+} f(\tau) & \text{if } t\in B\cup D.\end{cases}$$

Proof. This lemma is a direct extension of the previous lemma. We only need to show the first statement since the proof of the second is similar. We have

$$\left(\int_a^t f(\tau)\Delta\tau\right)^{\diamond_\alpha} = \alpha\left(\int_a^t f(\tau)\Delta\tau\right)^{\Delta} + (1-\alpha)\left(\int_a^t f(\tau)\Delta\tau\right)^{\nabla}$$
$$= \alpha f(t) + (1-\alpha)\begin{cases} f(\rho(t)) & \text{if } t \in B \cup C, \\ \lim_{\tau\to t^-} f(\tau) & \text{if } t \in A \cup D. \end{cases} \qquad \square$$

Based on the above, we state the following.

Theorem 3.5. *Let $a, t \in \mathbb{T}$. If the left-sided limits of $f : \mathbb{T} \to \mathbb{R}$ exist at left-dense points and the right-sided limits of the function exist at right-dense points in $\mathbb{T}$, then $\int_a^t f(\tau)\diamond_\alpha\tau$ is $\diamond_\alpha$-differentiable on $\mathbb{T}$, and*

$$\left(\int_a^t f(\tau)\diamond_\alpha\tau\right)^{\diamond_\alpha} = (1-2\alpha+2\alpha^2)f(t)$$
$$+\,\alpha(1-\alpha)\begin{cases} \lim_{\tau\to t^-} f(\tau) + f(\sigma(t)) & \textit{if } t \in A, \\ f(\rho(t)) + \lim_{\tau\to t^+} f(\tau) & \textit{if } t \in B, \\ f(\rho(t)) + f(\sigma(t)) & \textit{if } t \in C, \\ \lim_{\tau\to t^-} f(\tau) + \lim_{\tau\to t^+} f(\tau) & \textit{if } t \in D. \end{cases}$$

A natural consequence from [8] and the above theorem is the following.

Theorem 3.6. *Let $a, t \in \mathbb{T}$ and $f : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$. If*

(i) *f is* diamond-α *differentiable with respect to the first variable*;
(ii) *the left-sided limits of f exist at left-dense points and the right-sided limits of f exist at right-dense points with respect to the second valuable, then $\int_a^t f(t,\tau)\diamond_\alpha\tau$ is $\diamond_\alpha$-differentiable on $\mathbb{T}$, and*

$$\left(\int_a^t f(t,\tau)\diamond_\alpha\tau\right)^{\diamond_\alpha} = \int_a^t f^{\diamond_\alpha}(t,\tau)\diamond_\alpha\tau + \alpha^2 f(\sigma(t),t) + (1-\alpha)^2 f(\rho(t),t)$$
$$+\,\alpha(1-\alpha)\begin{cases} \lim_{\tau\to t^-} f(\rho(t),\tau) + f(\sigma(t),\sigma(t)) & \textit{if } t \in A, \\ f(\rho(t),\rho(t)) + \lim_{\tau\to t^+} f(\sigma(t),\tau) & \textit{if } t \in B, \\ f(\rho(t),\rho(t)) + f(\sigma(t),\sigma(t)) & \textit{if } t \in C, \\ \lim_{\tau\to t^-} f(\rho(t),\tau) + \lim_{\tau\to t^+} f(\sigma(t),\tau) & \textit{if } t \in D. \end{cases}$$

Proof. According to the definition,

$$
\begin{aligned}
&\left(\int_a^t f(t,\tau)\diamondsuit_\alpha\tau\right)^{\diamondsuit_\alpha}\\
&\quad=\alpha\left(\int_a^t f(t,\tau)\diamondsuit_\alpha\tau\right)^{\Delta}+(1-\alpha)\left(\int_a^t f(t,\tau)\diamondsuit_\alpha\tau\right)^{\nabla}\\
&\quad=\alpha\left(\alpha\int_a^t f(t,\tau)\Delta\tau+(1-\alpha)\int_a^t f(t,\tau)\nabla\tau\right)^{\Delta}\\
&\qquad+(1-\alpha)\left(\alpha\int_a^t f(t,\tau)\Delta\tau+(1-\alpha)\int_a^t f(t,\tau)\nabla\tau\right)^{\nabla}\\
&\quad=\alpha^2\int_a^t f^{\Delta}(t,\tau)\Delta t+(1-\alpha)^2\int_a^t f^{\nabla}(t,\tau)\nabla\tau\\
&\qquad+\alpha(1-\alpha)\left(\int_a^t f^{\Delta}(t,\tau)\nabla\tau+\int_a^t f^{\nabla}(t,\tau)\nabla\tau\right)\\
&\qquad+\alpha^2 f(\sigma(t),t)+(1-\alpha)^2 f(\rho(t),t)\\
&\qquad+\alpha(1-\alpha)\begin{cases}\lim\limits_{\tau\to t^-} f(\rho(t),\tau)+f(\sigma(t),\sigma(t)) & \text{if } t\in A,\\ f(\rho(t),\rho(t))+\lim\limits_{\tau\to t^+} f(\sigma(t),\tau) & \text{if } t\in B,\\ f(\rho(t),\rho(t))+f(\sigma(t),\sigma(t)) & \text{if } t\in C,\\ \lim\limits_{\tau\to t^-} f(\rho(t),\tau)+\lim\limits_{\tau\to t^+} f(\sigma(t),\tau) & \text{if } t\in D.\end{cases}\\
&\quad=\alpha\int_a^t f^{\Delta}(t,\tau)\diamondsuit_\alpha\tau+(1-\alpha)\int_a^t f^{\nabla}(t,\tau)\diamondsuit_\alpha\tau\\
&\qquad+\alpha^2 f(\sigma(t),t)+(1-\alpha)^2 f(\rho(t),t)\\
&\qquad+\alpha(1-\alpha)\begin{cases}\lim\limits_{\tau\to t^-} f(\rho(t),\tau)+f(\sigma(t),\sigma(t)) & \text{if } t\in A,\\ f(\rho(t),\rho(t))+\lim\limits_{\tau\to t^+} f(\sigma(t),\tau) & \text{if } t\in B,\\ f(\rho(t),\rho(t))+f(\sigma(t),\sigma(t)) & \text{if } t\in C,\\ \lim\limits_{\tau\to t^-} f(\rho(t),\tau)+\lim\limits_{\tau\to t^+} f(\sigma(t),\tau) & \text{if } t\in D.\end{cases}\\
&\quad=\int_a^t (\alpha f^{\Delta}(t,\tau)+(1-\alpha)f^{\nabla}(t,\tau))\diamondsuit_\alpha\tau\\
&\qquad+\alpha^2 f(\sigma(t),t)+(1-\alpha)^2 f(\rho(t),t)\\
&\qquad+\alpha(1-\alpha)\begin{cases}\lim\limits_{\tau\to t^-} f(\rho(t),\tau)+f(\sigma(t),\sigma(t)) & \text{if } t\in A,\\ f(\rho(t),\rho(t))+\lim\limits_{\tau\to t^+} f(\sigma(t),\tau) & \text{if } t\in B,\\ f(\rho(t),\rho(t))+f(\sigma(t),\sigma(t)) & \text{if } t\in C,\\ \lim\limits_{\tau\to t^-} f(\rho(t),\tau)+\lim\limits_{\tau\to t^+} f(\sigma(t),\tau) & \text{if } t\in D.\end{cases}.
\end{aligned}
$$

Thus the theorem is clear. □

Theorem 3.7. *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$, then*

(i) $\int_a^t [f(\tau) + g(\tau)] \diamondsuit_\alpha \tau = \int_a^t f(\tau) \diamondsuit_\alpha \tau + \int_a^t g(\tau) \diamondsuit_\alpha \tau$,
(ii) $\int_a^t cf(\tau) \diamondsuit_\alpha \tau = c \int_a^t f(\tau) \diamondsuit_\alpha \tau$,
(iii) $\int_a^t f(\tau) \diamondsuit_\alpha \tau = -\int_t^a f(\tau) \diamondsuit_\alpha \tau$,
(iv) $\int_a^t f(\tau) \diamondsuit_\alpha \tau = \int_a^b f(\tau) \diamondsuit_\alpha \tau + \int_b^t f(\tau) \diamondsuit_\alpha \tau$,
(v) $\int_a^a f(\tau) \diamondsuit_\alpha \tau = 0$.

Proof. The proofs are straightforward. As examples, here we only show (i) and (iv). For the former,

$$\begin{aligned}
\int_a^t [f(\tau) + g(\tau)] \diamondsuit_\alpha \tau &= \alpha \int_a^t [f(\tau) + g(\tau)] \Delta \tau + (1-\alpha) \int_a^t [f(\tau) + g(\tau)] \nabla \tau \\
&= \alpha \int_a^t f(\tau) \Delta \tau + (1-\alpha) \int_a^t f(\tau) \nabla \tau + \alpha \int_a^t g(\tau) \Delta \tau \\
&\quad + (1-\alpha) \int_a^t g(\tau) \nabla \tau \\
&= \int_a^t f(\tau) \diamondsuit_\alpha \tau + \int_a^t g(\tau) \diamondsuit_\alpha \tau.
\end{aligned}$$

As for (iv) we have

$$\begin{aligned}
\int_a^t f(\tau) \diamondsuit_\alpha \tau &= \alpha \int_a^t f(\tau) \Delta \tau + (1-\alpha) \int_a^t f(\tau) \nabla \tau \\
&= \alpha \int_a^b f(\tau) \Delta \tau + \alpha \int_b^t f(\tau) \Delta \tau + (1-\alpha) \\
&\quad \times \int_a^b f(\tau) \nabla \tau + (1-\alpha) \int_b^t f(\tau) \nabla \tau \\
&= \int_a^b f(\tau) \diamondsuit_\alpha \tau + \int_b^t f(\tau) \diamondsuit_\alpha \tau. \qquad \square
\end{aligned}$$

Corollary 3.8. *Let $t \in \mathbb{T}_\kappa^\kappa$, and $f, g : \mathbb{T} \to \mathbb{R}$, then*

$$\begin{aligned}
&\int_t^{\sigma(t)} f(\tau) \diamondsuit_\alpha \tau = \mu(t)(\alpha f(t) + (1-\alpha) f^\sigma(t)), \\
&\int_{\rho(t)}^t f(\tau) \diamondsuit_\alpha \tau = \eta(t)(\alpha f^\rho(t) + (1-\alpha) f(t)).
\end{aligned}$$

Proof. The results are direct generalizations of those by Messer [15] via an application of Theorem 3.7. $\square$

Theorem 3.9 (*Substitution Rule*). *Let $v : \mathbb{T} \to \mathbb{R}$ be monotone.*

(i) *Assume that v is strictly increasing and $\tilde{\mathbb{T}} = v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is a continuous function and v is $\diamond_\alpha$-differentiable, then for $a, t \in \mathbb{T}$,*

$$\int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\Delta\tau = \alpha\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\Delta}s + (1-\alpha)\int_a^t f(\tau)v^{\nabla}(\tau)\Delta\tau,$$

$$\int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\nabla\tau = \alpha\int_a^t f(\tau)v^{\Delta}(\tau)\nabla\tau + (1-\alpha)\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\nabla}s,$$

$$\begin{aligned}\int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\diamond_\alpha\tau &= \alpha\left(\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\Delta}s + \int_a^t f(\tau)v^{\Delta}(\tau)\nabla\tau\right)\\ &\quad + (1-\alpha)\left(\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\nabla}s + \int_a^t f(\tau)v^{\nabla}(\tau)\Delta\tau\right).\end{aligned}$$

(ii) *Assume that v is strictly decreasing and $\tilde{\mathbb{T}} = v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is a continuous function and v is $\diamond_\alpha$-differentiable, then for $a, t \in \mathbb{T}$,*

$$\int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\Delta\tau = \alpha\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\nabla}s + (1-\alpha)\int_a^t f(\tau)v^{\nabla}(\tau)\Delta\tau,$$

$$\int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\nabla\tau = \alpha\int_a^t f(\tau)v^{\Delta}(\tau)\nabla\tau + (1-\alpha)\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\Delta}s,$$

$$\begin{aligned}\int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\diamond_\alpha\tau &= \alpha\left(\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\nabla}s + \int_a^t f(\tau)v^{\Delta}(\tau)\nabla\tau\right)\\ &\quad + (1-\alpha)\left(\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\Delta}s + \int_a^t f(\tau)v^{\nabla}(\tau)\Delta\tau\right).\end{aligned}$$

Proof. We only show (i) since the proof of (ii) is similar. For this,

$$\begin{aligned}\int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\Delta\tau &= \alpha\int_a^t f(\tau)v^{\Delta}(\tau)\Delta\tau + (1+\alpha)\int_a^t f(\tau)v^{\nabla}(\tau)\Delta\tau\\ &= \alpha\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\Delta}s + (1-\alpha)\int_a^t f(\tau)v^{\nabla}(\tau)\Delta\tau.\\ \int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\nabla\tau &= \alpha\int_a^t f(\tau)v^{\Delta}(\tau)\nabla\tau + (1-\alpha)\int_a^t f(\tau)v^{\nabla}(\tau)\nabla\tau\\ &= \alpha\int_a^t f(\tau)v^{\Delta}(\tau)\nabla\tau + (1-\alpha)\int_{v(a)}^{v(t)} (f \circ v^{-1})(s)\tilde{\nabla}s.\\ \int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\diamond_\alpha\tau &= \alpha\int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\Delta\tau + (1-\alpha)\int_a^t f(\tau)v^{\diamond_\alpha}(\tau)\nabla\tau.\end{aligned}$$

The proof is thus completed by an application of the first two relations to the above equality. □

4. Computational experiments

It has been our intention to use combined dynamic derivatives for approximating derivative functions and solutions of nonlinear differential equations on time scales. The dynamic theory and methods on time scales have provided a useful tool in exploring certain important features of the algorithms involved. They may lead to more accurate and balanced computational procedures. The investigations are particularly interesting when nonuniform computational grids are adopted, on which conventional finite difference operators do not commute.

All our numerical experiments are carried out using MICROSOFT Visual C++, MATLAB subroutines and graphic packages on dual-processor DELL Precision workstations.

Example 1. We consider the first diamond-α derivative of the trigonometric function $y = \sin t$, $t \in [(\pi - 1)/2, (3\pi + 1)/2]$. The function is symmetric with respect to the center point in the interval. Our symmetric time scale $\mathbb{T}$ that superimposes the interval is

$$\begin{aligned}\{&0.9908, 1.5708, 2.1216, 2.3716, 2.6216, 2.7816, 2.9416, 3.0416, 3.1116, 3.1416,\\ &3.1716, 3.2416, 3.3416, 3.5016, 3.6616, 3.9116, 4.1616, 4.7124, 5.2924\}.\end{aligned} \tag{4.1}$$

Computed Δ, ∇ and $\diamondsuit_\alpha$ derivatives of the function ($\alpha = \frac{1}{2}$) over $\mathbb{T}^\kappa_\kappa$ are given in Fig. 1. It is observed that results of the diamond-α derivatives provide a surprisingly superior quality in approximating the classic derivative y' of the targeted trigonometric function with highly balanced accuracy. The phenomena are further verified by Fig. 2 where differences between the classical derivative $y' = \cos t$ and its approximations via Δ, ∇ dynamic derivatives and $\diamondsuit_\alpha$ derivative ($\alpha = \frac{1}{2}$) over the symmetric time scale $\mathbb{T}^\kappa_\kappa$ are given. The second frame of Fig. 3 gives a profile of the relative error of the combined dynamic derivative in ℓ_2 norm as α varies over [0, 1]. We find that the difference achieves its overall minimum as $\alpha \approx \frac{1}{2}$, while it becomes relatively large when $\alpha \approx 0$ or 1. It is also interesting to find that the error decays rapidly as α approaches $\frac{1}{2}$ in our particular example, and this indicates the usefulness of a combined derivative with nonzero α. The numerical experiments also suggest that the combined dynamic derivative may play a more important role in well-balanced function

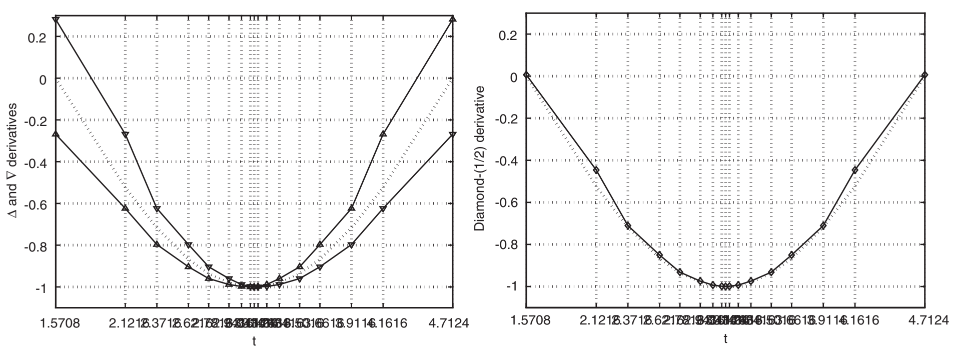

Fig. 1. Classic derivative y' (dot curve) and corresponding Δ, ∇ dynamic derivatives (LEFT), and the $\diamondsuit_\alpha$ derivative ($\alpha = \frac{1}{2}$, RIGHT).

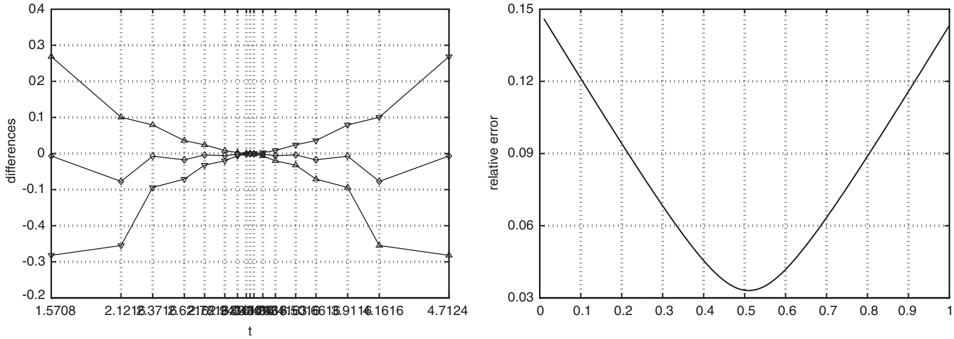

Fig. 2. Differences between the classic derivative and Δ, ∇ and $\diamond_\alpha$ derivatives ($\alpha = \frac{1}{2}$) on $\mathbb{T}_\kappa^\kappa$, respectively (LEFT); Profile of the relative error between the classic and the combined dynamic derivatives (ℓ_2 norm is used, RIGHT).

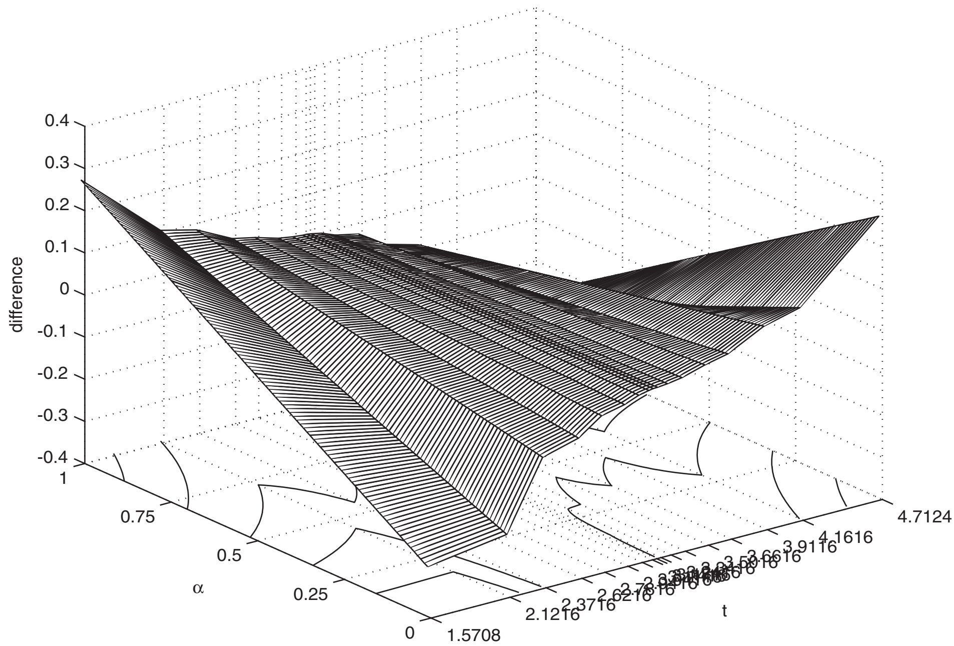

Fig. 3. Surface plot of the difference between the classic derivative $y' = \cos\, t$ and $\diamond_\alpha$ derivative on $\mathbb{T}_\kappa^\kappa$, $0 \leqslant \alpha \leqslant 1$.

approximations and in highlyaccurate finite difference schemes for solving differential equations. Finally, Fig. 7 provides a surface plot of the differences between the classic derivative $y' = \cos\, t$ and its diamond-α derivative on $\mathbb{T}_\kappa^\kappa$ as α varies from 0 to 1. The figure summarizes our discussions and comments on the higher quality of combined derivative approximations.

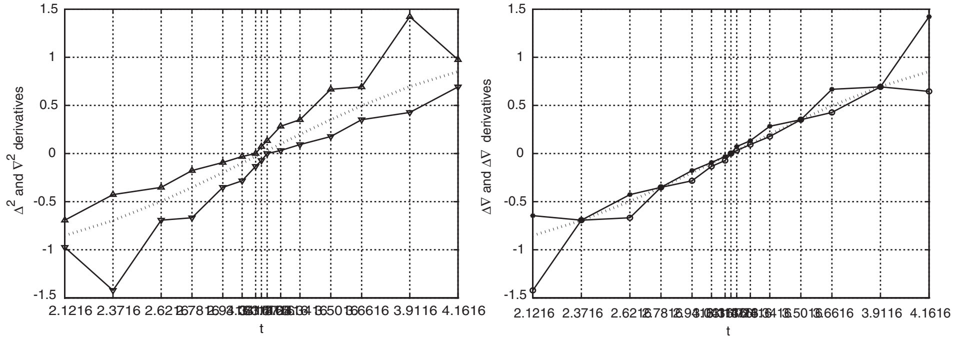

Fig. 4. Classic derivative y'' (dot curve) and $\varDelta^2$, ∇^2 dynamic derivatives (LEFT); and $\Delta\nabla$ (with circle marks), $\nabla\varDelta$ (with star marks) derivatives (RIGHT).

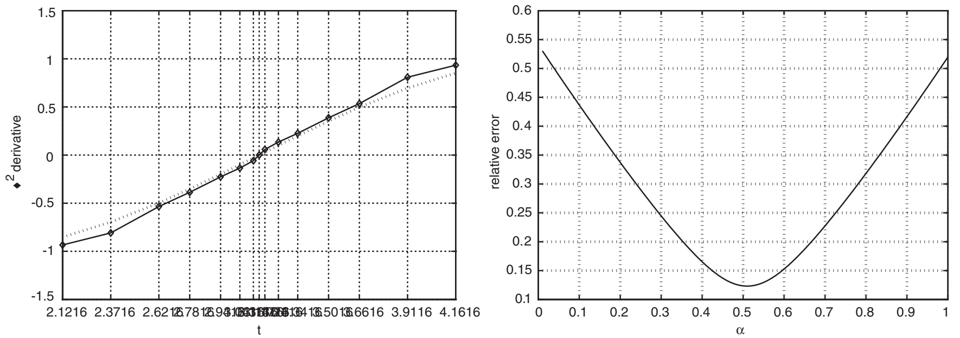

Fig. 5. Classic derivative y'' (dot curve) and the $\diamond_\alpha^2$ derivative (LEFT); Profile of the relative error between the classic and combined dynamic derivatives (ℓ_2 norm is used, RIGHT).

Example 2. We continue to work on the second diamond-α derivative of the trigonometric function $y = \sin\, t$, $t \in [(\pi - 1)/2,\ (3\pi + 1)/2]$. It is noticed that y'' is anti-symmetric [6] over the interval. The use of the symmetric time scale (4.1) is therefore justified. We assume $\mu(t)/\eta(t) = O(1)$, $t \in \mathbb{T}_\kappa^\kappa$, for the consistency.

Computed $\varDelta^2$, ∇^2, $\Delta\nabla$, $\nabla\varDelta$ dynamic derivatives and $\diamond_\alpha$ derivative ($\alpha = \frac{1}{2}$) of the targeted function over $\mathbb{T}_{\kappa^2}^{\kappa^2}$ are given in Figs. 4 and 5, respectively. It is observed while the pairs of dynamic derivatives ($\varDelta^2$, ∇^2) and ($\Delta\nabla$, $\nabla\varDelta$) exhibit correctly the anti-cross symmetric properties, their approximations to y'' are relatively poor, in particular due to the large graininess near the two ends of $\mathbb{T}$ used. However, even in this situation, the combined derivative demonstrates a superior quality in balanced approximation. Its accuracy is overwhelmingly superior as compared with its competitors. The second frame in Fig. 5 shows the profile of the relative error of the combined dynamic derivative in ℓ_2 norm as α varies within its full spectrum [0, 1]. It is very interesting to observe that the error decays dramatically as

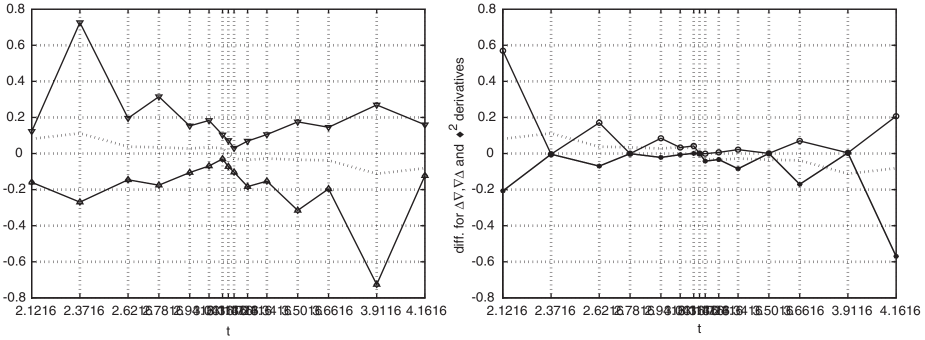

Fig. 6. Differences between the classic derivative and $\diamond_\alpha^2$ derivatives (dot curve, $\alpha = \frac{1}{2}$), and Δ^2, ∇^2 dynamic derivatives, respectively (LEFT); and the $\Delta\nabla$ (with circle marks), $\nabla\Delta$ (with star marks), respectively (RIGHT).

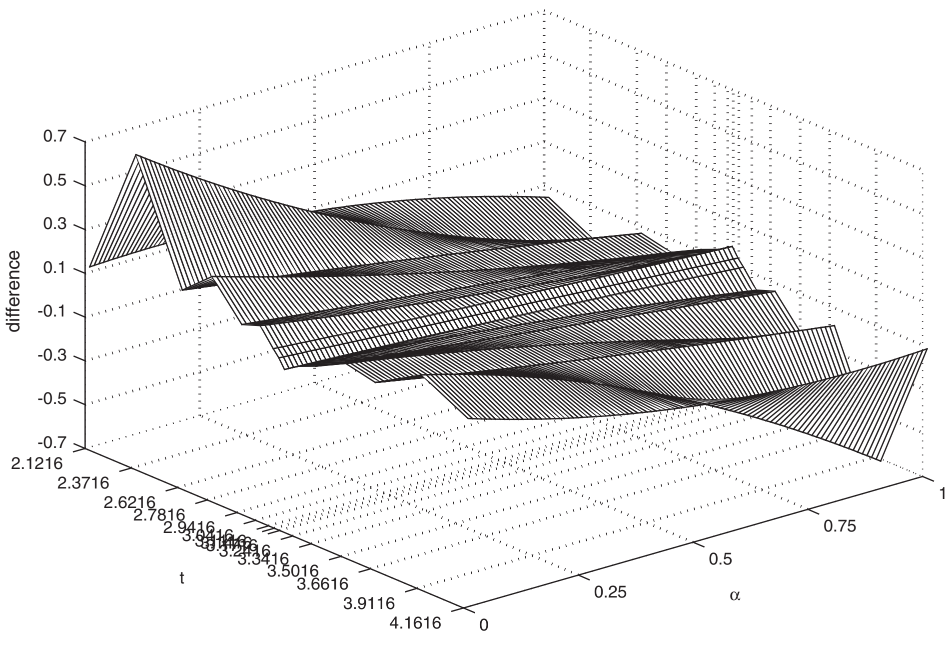

Fig. 7. Surface plot of the difference between classic derivative $y'' = -\sin t$ and the $\diamond_\alpha^2$ derivative on $\mathbb{T}_{\kappa^2}^{\kappa^2}$, $0 \leqslant \alpha \leqslant 1$.

α approaches $\frac{1}{2}$ in this particular example, and this implies the usefulness of a combined derivative with nontrivial α employed.

The conclusions are further confirmed by Fig. 6 where differences between the classical derivative $y'' = -\sin t$ and its approximations via Δ^2, ∇^2, $\Delta\nabla$, $\nabla\Delta$ dynamic derivatives and the $\diamond_\alpha^2$ derivative ($\alpha = \frac{1}{2}$) over the symmetric time scale $\mathbb{T}_{\kappa^2}^{\kappa^2}$ are given. Finally, Fig. 7

provides a surface plot of the differences between the classic derivative $y'' = -\sin t$ and the combined derivative as α changes from 0 to 1. It is again observed that the difference achieves its overall minimum when $\alpha \approx \frac{1}{2}$, while it becomes relatively large as α is about 0 or 1.

Example 3. Many important mathematical models in science and engineering applications can be reduced to nonhomogeneous two-point boundary problems [6,9,16,17]. As an example, we consider the following scalar problem:

$$u''(t) = f(t), \quad t \in (a, b), \tag{4.2}$$

$$u(a) = 0, \quad u(b) = 0, \tag{4.3}$$

where $a = \pi/2$, $b = 3\pi/2$.

Set $f(t) = 1$. We consider the solution of a diamond-α dynamic boundary value problem that can be viewed as an approximation of (4.2) and (4.3). We assume that the graininess ratio $\mu(t)/\eta(t)$ is approximately one for a good consistency [16,17]. The analytic solution of (4.2), (4.3) is $u(t) = (ab - (a+b)t + t^2)/2$ which is symmetric with respect to the midpoint of the interval $[a, b]$. To avoid any auxiliary boundary conditions and improve the solvability of the discrete algorithm, based on Theorem 2.4, we introduce minor perturbations in the combined dynamic derivative and consider the following dynamic boundary value problem,

$$\alpha^2 u^{\Delta\Delta}(t^{\rho}) + \alpha(1-\alpha)(u^{\Delta\nabla}(t) + u^{\nabla\Delta}(t)) + (1-\alpha)^2 u^{\nabla\nabla}(t^{\sigma}) = 1, \quad t \in \mathbb{T}_{\kappa}^{\kappa}, \tag{4.4}$$

$$u(a) = 0, \quad u(b) = 0, \quad 0 \leqslant \alpha \leqslant 1, \tag{4.5}$$

where the symmetric time scale

$$\begin{aligned}\mathbb{T} = \{&1.57079632, 1.87079632, 2.17079632, 2.42089632, 2.62139632,\\ &2.78159632, 2.91159632, 3.01159632, 3.08159632, 3.12159632,\\ &3.14159632, 3.16159632, 3.20159632, 3.27159632, 3.37159632,\\ &3.50159632, 3.66179632, 3.86229632, 4.11239632,\\ &4.41239632, 4.71239632\}\end{aligned}$$

which is built via the graininess distribution,

$$\begin{aligned}\mathscr{G} = \{&0.3, 0.3, 0.2501, 0.2005, 0.1602, 0.13, 0.1, 0.07, 0.04, 0.02, 0.02,\\ &0.04, 0.07, 0.1, 0.13, 0.1602, 0.2005, 0.2501, 0.3, 0.3\}.\end{aligned}$$

In Fig. 8, we show the numerical solution of the dynamic boundary value problem (4.4), (4.5). In the first frame, solutions for $\alpha = 0, 1$, respectively, are given. Since the two cases are equivalent to that using $\nabla\nabla$, $\Delta\Delta$ derivatives, respectively, the solutions exhibit clear cross-symmetric properties [6–8]. On the other hand, when $\alpha = \frac{1}{2}$, the numerical solution obtained shows excellent symmetry and accuracy in approximations. We may notice that relatively large error appears around the center of the time scales and this is probably due to the shifts used in (4.4) for achieving the simplicity in calculations. Fig. 9 gives the differences and errors of the numerical solutions, respectively. In the left frame, differences between the

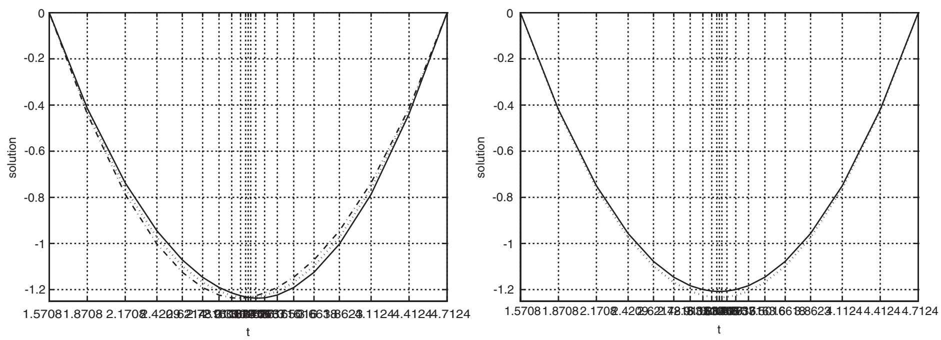

Fig. 8. The exact (dot curve) and dynamic solutions when $\alpha = 0, 1$ (the latter is in a dash-dot curve, LEFT); and that when ($\alpha = \frac{1}{2}$, RIGHT).

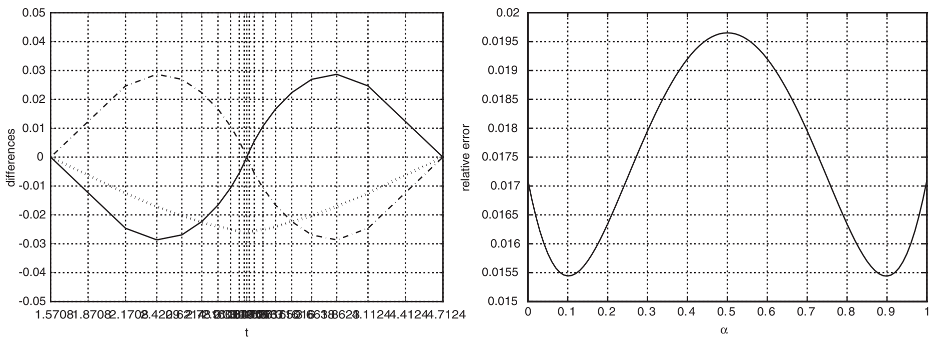

Fig. 9. Differences between the exact and dynamic solutions as $\alpha = \frac{1}{2}$ (dot curve), $\alpha = 0$ (solid curve), and $\alpha = 1$ (dash-dot curve, LEFT); and the relative error profile (ℓ_2 norm is used, RIGHT).

exact solution of (4.2), (4.3) and solutions of dynamic boundary value problem as $\alpha = 0$ (solid curve), $\alpha = 1$ (dash-dot curve), and $\alpha = \frac{1}{2}$ (dot curve) are given. The experiments again confirm the cross-symmetry for the first two solutions, and suggest that the solution with nontrivial α is much superior in the sense of providing a more balanced numerical approximation.

Further, the second frame of Fig. 9 provides the relative error profile of u as α varies between 0 and 1. It can be observed that the relative error is small throughout computations, especially when $\alpha \approx 0.1$ or 0.9. Of course, it is understood that the least relative error may not indicate the best symmetric pattern preservation. However, it may demonstrate a good approximation overall. Fig. 10 gives a 3-dimensional surface plot of the difference between the solution of (4.2), (4.3) and (4.4), (4.5). The pattern of the surface agrees well with our previous conclusions, and suggests that the use of combined dynamic derivatives is not trivial in computational applications.

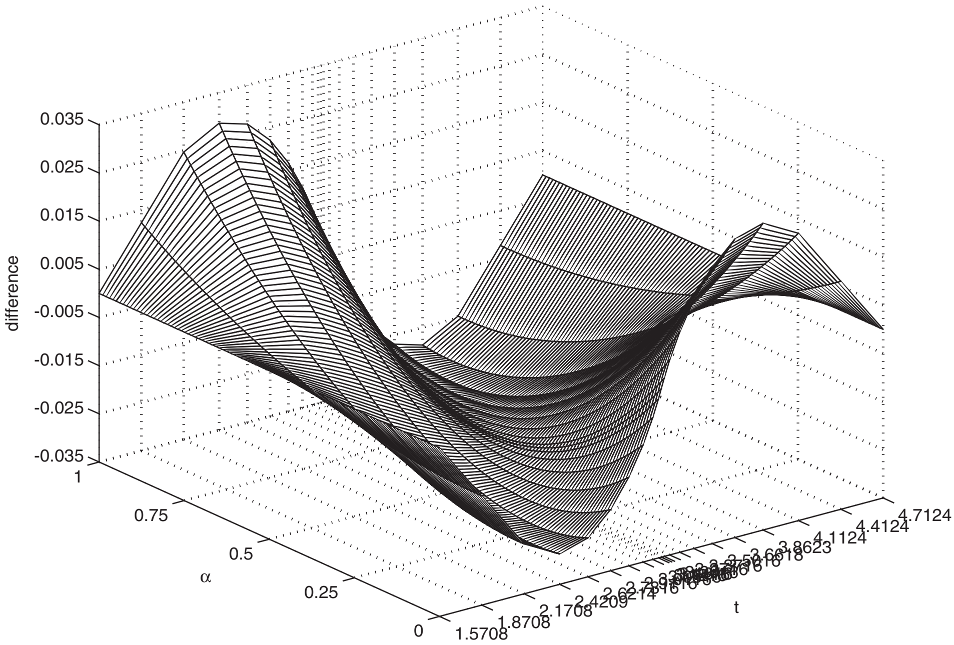

Fig. 10. Surface plot of the difference between exact solution and the dynamic solution on $\mathbb{T}$, $0 \leqslant \alpha \leqslant 1$.

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