ABSTRACT

Lyapunov Stability and Floquet Theory for
Nonautonomous Linear Dynamic Systems on Time Scales

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In this work, the stability of nonautonomous linear dynamic systems on time scales is investigated and analyzed. A unified and extended version of Lyapunov's direct method is developed and yields criteria for uniform stability and uniform exponential stability of a linear dynamic system. We investigate "slowly varying" nonautonomous systems and provide a spectral condition on the system matrix sufficient for exponential stability. Perturbations of the unforced system are studied and an instability criterion is introduced. We develop a comprehensive, unified Floquet theory including Lyapunov transformations and their various stability preserving properties, as well as a unified Floquet theorem which establishes a canonical Floquet decomposition on time scales in terms of the generalized exponential function. We then use these results to study homogenous as well as nonhomogeneous periodic problems. Furthermore, we explore the connection between Floquet multipliers and Floquet exponents via monodromy operators and establish a spectral mapping theorem on time scales. We conclude with several nontrivial examples to show the utility of this theory.

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DEDICATION

To the most beautiful, selfless, caring, intelligent, strong,

and amazing person that I have ever met in my life.

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You are a wonderful, remarkable woman. Thank you so much for being you.

CHAPTER ONE

An Introduction and Overview

1.1 Unification and Extension

In 1988, Stephan Hilger's Ph.D. thesis [24] introduced the theory of time scales for the purpose of unifying discrete and continuous analysis. By developing a theory for "dynamic equations" on very general domains (time scales), one can produce a more general result that can then be applied to the desired domain which can be a hybrid discrete/continuous domain.

There are many results from differential equations that carry over quite naturally and easily to difference equations, while others have a completely different structure from their continuous counterparts. The study of dynamic equations on time scales sheds new light on the discrepancies between continuous differential equations and discrete difference equations. It also prevents one from proving a result twice, once for differential equations and once for difference equations. The general idea, which is the main goal of Bohner and Peterson's excellent introductory text [6], is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale.

One can choose the time scale to be the set of the reals. The general result obtained yields the same result concerning an ordinary differential equation. One can also choose the time scale to be the set of integers. The general result is the same result one would obtain concerning a difference equation. However, since there are infinitely many other time scales that one may work with besides the reals and the integers, one has a much more general result. Thus, the two main features of the time scales calculus are *unification* and *extension*.

In Chapter 2, the time scales calculus is developed. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals. For functions $f : \mathbb{T} \to \mathbb{R}$ a derivative and an integral are introduced. Fundamental results, e.g., the product rule and the quotient rule, are presented. Other results concerning differentiability and integrability are stated, as they will be necessary for the subsequent portion of this dissertation.

The Hilger complex plane is introduced, along with the cylinder and inverse cylinder transformations. The cylinder transformation is used to develop the generalized exponential function on time scales, defined as the solution of a first order dynamic initial value problem. Properties of the time scale exponential function are developed. For the nonhomogeneous case, a generalized form of the variation of constants is used.

Uniqueness and existence theorems are also presented and the matrix exponential function on a time scale is introduced. Properties of the transition matrix and the matrix exponential function are stated and used throughout.

In Chapter 3, general definitions are provided concerning the matrix norms used throughout the dissertation as well as the notion of definiteness of a matrix. The concepts of uniform stability, uniform exponential stability, and uniform asymptotic stability are also defined. Several theorems which characterize the stability of the system with respect to the transition matrix are stated and proved.

1.2 Lyapunov Stability Theory

It is widely known that the stability characteristics of an autonomous linear system of differential or difference equations can be characterized completely by the placement of the eigenvalues of the system matrix [3, 22]. Recently, Pötzsche, Siegmund, and Wirth [40] authored a landmark paper which developed necessary and sufficient conditions for the stability of time invariant linear systems on arbitrary time scales. Their characterization included the sufficient condition that the eigenvalues of the system matrix be contained in the possibly disconnected set of stability $\mathcal{S}(\mathbb{T}) \subset \mathbb{C}^-$, which may change for each time scale on which the system is studied. The subsequent paper by Hoffacker and Gard [16] further examined the stability characteristics of time varying and time invariant scalar dynamic equations on time scales. This is the first paper to characterize the behavior of a time varying first order dynamic equation on arbitrary time scales.

The intent of Chapter 4 is to extend the current results of autonomous linear dynamic systems to the more general case of nonautonomous linear dynamic systems on a large class of time scales (i.e. those time scales with bounded graininess which are unbounded above). We show that, in general, the placement of eigenvalues of the system matrix does *not* guarantee the stability or exponential stability of the time varying system, as is the case with autonomous linear systems of differential and difference equations [7, 22, 30, 31, 42] and certain dynamic equations on time scales [40]. We unify and extend the theorems of eigenvalue placement in the proper region of the complex plane for sufficiently slowly-varying system matrices of continuous and discrete nonautonomous systems, which yields exponential stability of the system, as in the classic papers of Desoer [13, 14], Rosenbrock [41], and the relatively recent paper by Solo [46]. To develop this theory for nonautonomous systems, we unify the theorems of uniform stability, uniform exponential stability, and uniform asymptotic stability for time varying systems by implementing a generalized time scales version of the direct (second) method of Lyapunov [35], as in the standard papers on stability of continuous and discrete dynamical systems by Kalman and Bertram [30, 31].

In his dissertation of 1892, Lyapunov developed two methods for analyzing the stability of differential equations. His direct method has become the most widely used tool for stability analysis of linear and nonlinear systems in both differential and difference equations. The idea involves measuring the energy of the system, usually the norm of the state variables, as the system evolves in time. The objective of the approach is the following: *To answer questions of stability of differential and difference equations, utilizing the given form of the equations but without explicit knowledge of* *the solutions.* The principal idea of the direct method is contained in the following physical reasoning: If the rate of change, $dE(x)/dt$, of the energy $E(x)$ of an isolated physical system is negative for every possible state x, except for a single equilibrium state x_e, then the energy will continually decrease until it finally assumes its minimum value $E(x_e)$. In other words, a system that is perturbed from its equilibrium state will always return to it. This is the intuitive concept of stability. It follows that the mathematical counterpart of the preceding statement is the following: *A dynamic system is stable* (*in the sense that it returns to equilibrium after any perturbation*) *if and only if there exists a "Lyapunov function," i.e., some scalar function* $V(x)$ *of the state with the properties:* (*a*) $V(x) > 0$, $\dot{V}(x) < 0$ *when* $x \neq x_e$, *and* (*b*) $V(x) = \dot{V}(x) = 0$ *when* $x = x_e$.

In engineering applications and applied mathematics problems, a solution usually is not readily available nor easily calculated. As in adaptive control, which was born from a desire to stabilize certain classes of continuous linear systems without the need to explicitly identify the unknown system parameters, even a knowledge of the system matrix itself may not be fully available. The inherent beauty and elegance of the direct method of Lyapunov is that knowledge of the exact solution is not necessary. The qualitative behavior of the solution to the system (i.e. stability or instability) can be investigated without computing the actual solution.

By unifying and extending Lyapunov's direct method to nonautonomous linear systems on time scales, we encounter the possibility of a time domain consisting of nonuniform distance between successive points. This proves to be a nontrivial issue and hence is seldom dealt with in the literature. It is, however, a rapidly increasing theme in many engineering applications, such as the papers by Ilchmann, Owens, and Prätzel-Wolters [26], Ilchmann and Ryan [27], and Ilchmann and Townley [28], which deal with high gain adaptive controllers and digital systems, as well as the very recent results from Gravagne, Davis, DaCunha, and Marks [20, 21] which give new algorithms for adaptive controllers and bandwidth reduction using controller area networks via nonuniform sampling. The time scale methods introduced and developed in this paper allow the examination and analyzation of the stability characteristics of dynamical systems without regard to the particular domain of the system, i.e. continuous, discrete, or hybrid.

In Section 4.1, the general idea of the stability of a system is investigated and a quadratic Lyapunov function is developed for use in the remaining part of the chapter. Sections 4.2 and 4.3 introduce the unified theorems of uniform stability and uniform exponential stability of linear dynamic systems on time scales, as well as illustrations of these theorems in examples. The generalized Lyapunov matrix equation on a time scale is introduced in Section 4.4 and a closed form solution is given. Section 4.5 gives conditions on the eigenvalues of a sufficiently "slowly varying" system matrix which ensures exponential stability of the system solution. In Section 4.6, the stability properties of systems with linear and nonlinear perturbations are investigated. Finally, Section 4.7 demonstrates how the quadratic Lyapunov function developed in Section 4.1 can also be used to determine the instability of a system.

1.3 The Lyapunov Transformation and Floquet Theory

One of the many applications of the Lyapunov transformation of variables includes generating different state variable descriptions of linear time invariant systems because different state variable descriptions correspond to different and perhaps more advantageous points of view in determining the system's output characteristics. This is useful in signals and systems applications for the simple fact that different descriptions of state variables allow usage of linear algebra to design and study the internal structure of a system. Having the ability to change the internal structure without changing the input-output behavior of the system is useful for identifying implementations of these systems that optimize some performance criteria that may not be directly related to input-output behavior, such as numerical effects of round-off error in a computer-based systems implementation. For example, using a transformation of variables on a discrete time nondiagonal 2×2 system, one can obtain a diagonal system matrix which separates the state update into two decoupled first-order difference equations, and, because of its simple structure, this form of the state variable description is very useful for analyzing the system's properties [23].

The stability characteristics of a nonautonomous periodic linear system of differential or difference equations can be characterized completely by a corresponding autonomous linear system of differential or difference equations by a periodic Lyapunov transformation of variables [10, 32, 42]. Without question, the study of periodic systems in general and Floquet theory in particular has been central to the differential equations theorist for some time. Researchers have explored these topics for ordinary differential equations [10, 15, 17, 29, 38, 39, 45, 48], partial differential equations [8, 11, 17, 33], differential-algebraic equations [12, 34], and discrete dynamical systems [1, 32, 47]. Certainly [36] is a landmark paper in the area. Not surprisingly, Floquet theory has wide ranging effects, including extensions from time varying linear systems to time varying nonlinear systems of differential equations of the form $x' = f(t, x)$, where $f(t, x)$ is smooth and ω-periodic in t. The paper by Shi [44] ensures the global existence of solutions and proves that this system is topologically equivalent to an autonomous system $y' = g(y)$ via an ω-periodic transformation of variables. The theory has also been extended by R. Weikard [48] to nonautonomous linear systems of the form $\dot{z} = A(x)z$ where $A : \mathbb{C} \to \mathbb{C}^{n \times n}$ is an ω-periodic function in the complex variable x, whose solutions are meromorphic. With the assumption that $A(x)$ is bounded at the ends of the period strip, it is shown that there exists a fundamental solution of the form $P(x)e^{Jx}$ with a certain constant matrix J and function P which is rational in the variable $e^{2\pi ix/\omega}$.

In a relatively recent paper by Teplinskiĭ and Teplinskiĭ [47], Lyapunov transformations and discrete Floquet theory are extended to countable systems in $l_\infty(\mathbb{N}, \mathbb{R})$. It is proved that the countable time varying system can be represented by a countable time invariant system provided its finite-dimensional approximations can also be represented by time invariant systems.

Lyapunov transformations and Floquet theory have also been used to analyze the stability characteristics of quasilinear systems with periodically varying parameters. In 1994, Pandiyan and Sinha [38] introduced a new technique for the investigation of these systems based on the fact that all quasilinear periodic systems can be replaced by similar systems whose linear parts are time-invariant, via the well known Lyapunov-Floquet transformation.

In the paper by Demir [12], the equivalent of Floquet theory is developed for periodically time-varying systems of linear DAEs: $\frac{d}{dt}(C(t)x) + G(t)x = 0$ where the $n \times n$ matrices $C(\cdot)$ (not full rank in general) and $G(\cdot)$ are periodic. This result is developed for a direct application to oscillators which are ubiquitous in physical systems: gravitational, mechanical, biological, and especially electronic and optical ones. For example, in radio frequency communication systems, they are used for frequency translation of information signals and for channel selection. Oscillators are also present in digital electronic systems which require a time reference, i.e., a clock signal, in order to synchronize operations. All physical systems, and in particular electronic ones, are corrupted by undesired perturbations such as random thermal noise, substrate and supply noise, etc. Hence, signals generated by practical oscillators are not perfectly periodic. This performance limiting factor in electronic systems is also analyzed in [12] and a theory for nonlinear perturbation analysis of oscillators described by a system of DAEs is developed.

In this dissertation, we extend the current results of continuous and discrete Floquet theory to the more general case of an arbitrary periodic time scale, which will be defined in a subsequent chapter. In particular, one of the main results shows that if there exists an $n \times n$ constant matrix R such that $e_R(t_0+p, t_0) = \Phi_A(t_0+p, t_0)$ (where $\Phi_A(t, t_0)$ is the transition matrix for the p-periodic system $x^{\Delta}(t) = A(t)x(t)$, $x(t_0) = x_0$ and $e_R(t, t_0)$ is the time scale matrix exponential), then the transition matrix can be represented by the product of a p-periodic Lyapunov transformation matrix and a time scale matrix exponential, i.e. $\Phi_A(t, t_0) = L(t)e_R(t, t_0)$, which is known as the *Floquet decomposition* of the transition matrix $\Phi_A(t, t_0)$.

There has been one attempt at generalizing the Floquet decomposition to the time scales case by Ahlbrandt and Ridenhour [1]. However, there are some important distinctions between that work and this one. First, Ahlbrandt and Ridenhour use a different definition of a periodic time scale. Furthermore—and very importantly—their Floquet decomposition theorem employs the usual exponential function whereas our approach is more general (and we think more appropriate) since it is in terms of the generalized time scale exponential function. Finally, we go on to develop a complete Floquet theory including Lyapunov transformations and their stability preserving properties, Floquet multipliers, and Floquet exponents.

We also mention that the notion of a generalized time scale matrix logarithm remains an open question. If the existence of this logarithm can be shown, then the question of the existence of a solution matrix M to the matrix equation $e_M(t, \tau) = N$, where M and N are $n \times n$ matrices, will be confirmed and the calculation of such a matrix M will be greatly simplified. As of now, there is no general method or closed form of the solution matrix M in the general time scale case.

In Chapter 5 the generalized Lyapunov transformation for time scales is developed and it is shown that the change of variables using the time scales version of this transformation preserves the stability properties of the system.

In Chapter 6, the notion of a periodic time scale is presented and the main theorem, the unified and extended version of the Floquet decomposition theorem, is introduced for the homogeneous and nonhomogeneous cases of a periodic system on a periodic time scale. Three examples are given in Section 6.3 to illustrate how the unified Floquet theory applies in the cases $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and more interestingly, when $\mathbb{T} = \mathbb{P}_{1,1}$.

Chapter 7 introduces unified theorems involving Floquet multipliers, Floquet exponents, as well as a generalized spectral mapping theorem for time scales.

In Chapter 8, the examples from Section 6.3 are revisited and the theorems introduced in Chapter 7 are illustrated.

CHAPTER TWO

The Calculus of Time Scales

The following definitions and theorems, as well as a general introduction to the theory, can be found in the text by Bohner and Peterson [6].

Definition 2.1. A *time scale* $\mathbb{T}$ is any closed subset of $\mathbb{R}$.

Definition 2.2. The *forward jump operator*, $\sigma(t)$, and the *backward jump operator*, $\rho(t)$, are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

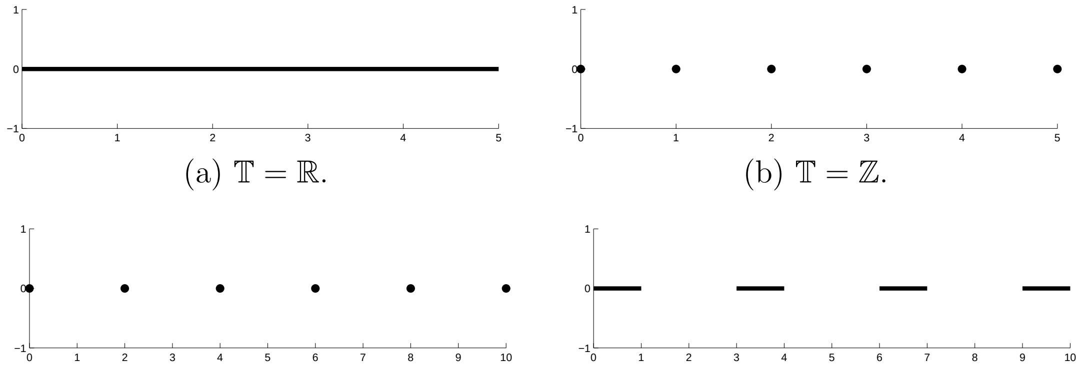
Definition 2.3. An element $t \in \mathbb{T}$ is *left-dense, right-dense, left-scattered, right-scattered* if $\rho(t) = t$, $\sigma(t) = t$, $\rho(t) < t$, $\sigma(t) > t$, respectively. Also, $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.4. The distance from an element $t \in \mathbb{T}$ to its successor is called the *graininess* of t and is denoted by $\mu(t) = \sigma(t) - t$.

We remark that in this dissertation, we denote the maximum graininess of a time scale as $\mu_{\max} = \sup_{t\in\mathbb{T}} \mu(t)$ and the maximum delta derivative of the graininess as $\mu^\Delta_{\max} = \sup_{t\in\mathbb{T}} \mu^\Delta(t)$.

2.1 Examples of Time Scales

Example 2.1. For $\mathbb{T} = \mathbb{R}$ we have $\sigma(t) = t = \rho(t)$ and $\mu(t) = 0$. For $\mathbb{T} = \mathbb{Z}$ we have $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) = 1$. See Figure 2.1(a) and (b).



(a) $\mathbb{T} = \mathbb{R}$. (b) $\mathbb{T} = \mathbb{Z}$.

(c) $\mathbb{T} = 2\mathbb{Z}$. (d) $\mathbb{T} = \mathbb{P}_{1,2}$.

Figure 2.1. Some canonical time scales.

Example 2.2. Let $h > 0$ be a fixed real number. Define the time scale $h\mathbb{Z}$ by

$$h\mathbb{Z} = \{hz : z \in \mathbb{Z}\} = \{\dots, -3h, -2h, -h, 0, h, 2h, 3h, \dots\}.$$

Here, $\sigma(t) = t + h$, $\rho(t) = t - h$, and $\mu(t) = h$. See Figure 2.1(c).

Example 2.3. Let $a, b > 0$ be fixed real numbers. Define the time scale $\mathbb{P}_{a,b}$ by

$$\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a];$$

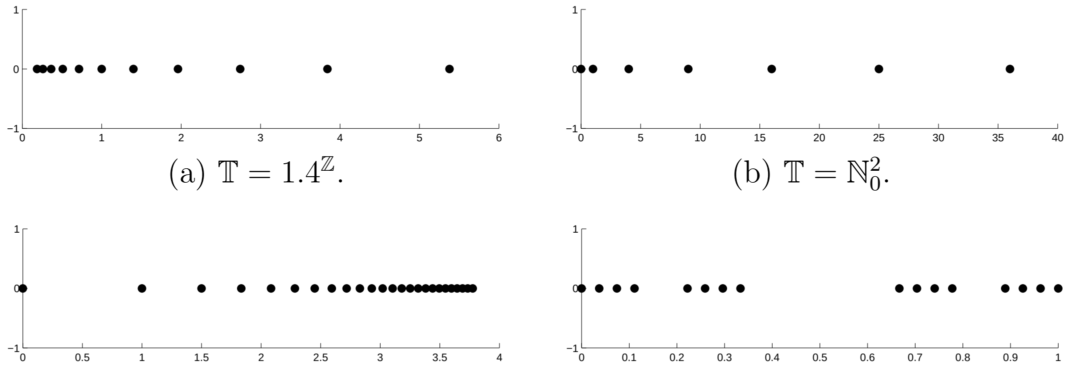
that is, $\mathbb{P}_{a,b}$ is a collection of closed intervals anchored at 0, each with length a and gap length between intervals being b. See Figure 2.1(d). Easy calculations show

$$\sigma(t) = \begin{cases} t, & t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a), \\ t + b, & t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\}, \end{cases}$$

$$\rho(t) = \begin{cases} t, & t \in \bigcup_{k=0}^{\infty} (k(a+b), k(a+b)+a], \\ t - b, & t \in \bigcup_{k=1}^{\infty} \{k(a+b)\}, \end{cases}$$

and

$$\mu(t) = \begin{cases} 0, & t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a), \\ b, & t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\}. \end{cases}$$



(a) $\mathbb{T} = 1.4^{\mathbb{Z}}$. (b) $\mathbb{T} = \mathbb{N}_0^2$.

(c) $\mathbb{T} = \mathbb{H}$. (d) $\mathbb{T} = \mathbb{K}_3$.

Figure 2.2. More protypical time scales.

Example 2.4. Let $q > 1$ be a fixed real number. Define the time scale $q^{\mathbb{Z}}$ by

$$q^{\mathbb{Z}} = \{q^z : z \in \mathbb{Z}\} = \{\dots, q^{-3}, q^{-2}, q^{-1}, 1, q, q^2, q^3, \dots\}.$$

Here, $\sigma(t) = qt$, $\rho(t) = t/q$, and $\mu(t) = (q-1)t$ for any t in this time scale. See Figure 2.2(a). We can then define the similar time scale

$$\overline{q^{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}.$$

Note that every nonzero point in $\overline{q^{\mathbb{Z}}}$ is isolated, but zero itself is right-dense.

Example 2.5. Define the time scale $\mathbb{N}_0^2$ by

$$\mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\} = \{0, 1, 4, 9, 16, \dots\}.$$

Here, $\sigma(t) = t + 2\sqrt{t} + 1$, $\rho(t) = t - 2\sqrt{t} + 1$, and $\mu(t) = 1 + 2\sqrt{t}$ for any t in this time scale. See Figure 2.2(b).

Example 2.6. Let $n \in \mathbb{N}_0$. Define the *harmonic numbers* H_n recursively by

$$H_0 = 0 \qquad \text{and} \qquad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Then define the time scale

$$\mathbb{H} = \{H_n : n \in \mathbb{N}_0\}.$$

For this time scale, $\sigma(H_n) = \displaystyle\sum_{k=1}^{n+1} \frac{1}{k}$, while

$$\rho(H_n) = \begin{cases} \displaystyle\sum_{k=1}^{n-1} \frac{1}{k}, & n \geq 2, \\ 0, & n = 0, 1, \end{cases}$$

and $\mu(H_n) = \frac{1}{n+1}$. See Figure 2.2(c).

Example 2.7. A more exotic example of a time scale is the Cantor set which is constructed as follows. Let $K_0 = [0, 1]$. To obtain K_1, remove the open middle third of the previous interval to get $K_1 = [0, 1/3] \cup [2/3, 1]$. To get K_2, remove the open middle-thirds from each subinterval in K_1 so that $K_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Continue in this way indefinitely. The Cantor set $\mathbb{K}$ is defined as

$$\mathbb{K} = \bigcap_{n=0}^{\infty} K_n.$$

In other words, after continuing the process above indefinitely, the Cantor set is all the points in $[0, 1]$ that do not get removed (i.e., they must be the endpoints of a subinterval in some K_n). See Figure 2.2(d) and Figure 2.3.

2.2 Differentiation

Definition 2.5. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, define $f^\Delta(t)$, the *delta derivative* of $f(t)$, as the number (when it exists), with the property that, for any $\epsilon > 0$, there exists a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

If f is delta differentiable for every $t \in \mathbb{T}^\kappa$, then $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* on $\mathbb{T}^\kappa$. We say f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is called the *delta derivative* of f on $\mathbb{T}^\kappa$.

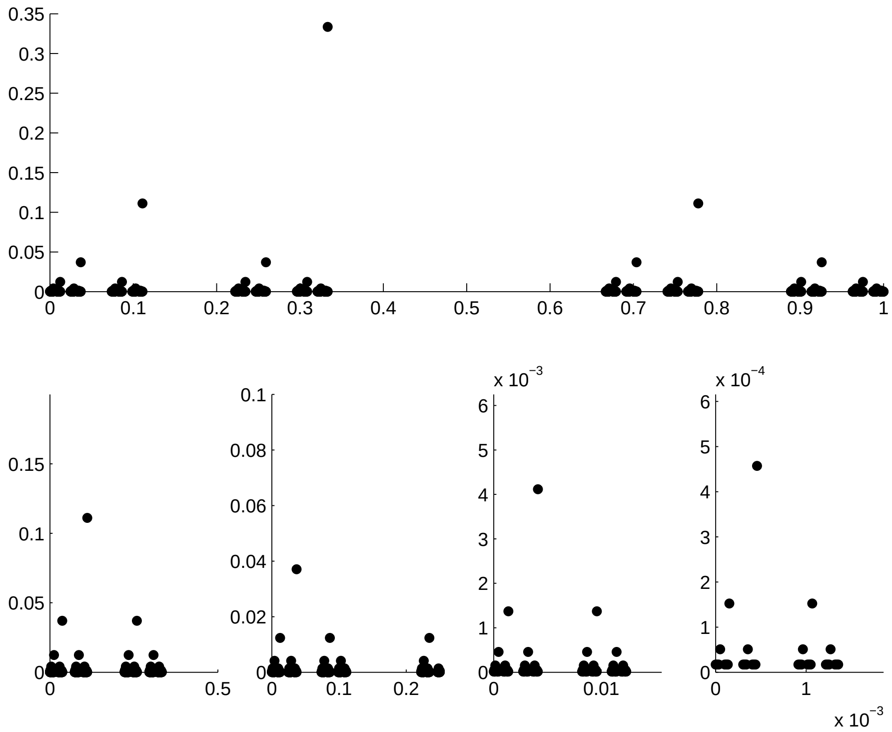


Figure 2.3. The plot of $\mu(t)$ on $\mathbb{K}$ exhibits fractal characteristics.

Theorem 2.1. [6] *Suppose* $f : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^\kappa$*.*

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is delta differentiable at t and $f^\Delta(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$.

(iii) If t is right-dense, then f is delta differentiable at t if and only if $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ *exists. In this case,* $f^\Delta(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$*.*

Note that f^Δ is precisely f' from the usual calculus when $\mathbb{T} = \mathbb{R}$. On the other hand, $f^\Delta = \Delta f = f(t+1) - f(t)$ (i.e. the forward difference operator) on the time scale $\mathbb{T} = \mathbb{Z}$. These are but two very special (and rather simple) examples of time scales. Moreover, the realms of differential equations and difference equations can now be viewed as but special, particular cases of more general *dynamic equations on time scales*, i.e. equations involving the delta derivative(s) of some unknown function.

Table 2.1. Basic notions from time scales calculus.

$\mathbb{T}=\mathbb{R}$	$\mathbb{T}=\mathbb{Z}$	Any $\mathbb{T}$
$(k\,f)' = k \cdot f'$	$\Delta(k\,f) = k\Delta f$	$(k\,f)^{\Delta} = k \cdot f^{\Delta}$
$(f+g)' = f' + g'$	$\Delta(f+g) = \Delta f + \Delta g$	$(f+g)^{\Delta} = f^{\Delta} + g^{\Delta}$
$(fg)' = fg' + f'g$	$\Delta(fg) = f\Delta g + \Delta f \cdot g(t+1)$	$(fg)^{\Delta} = f \cdot g^{\Delta} + f^{\Delta} \cdot g^{\sigma}$
$(f/g)' = \frac{f'g - fg'}{g^2}$	$\Delta(f/g) = \frac{\Delta f \cdot g - f\Delta g}{g \cdot g(t+1)}$	$(f/g)^{\Delta} = \frac{f^{\Delta}g - f \cdot g^{\Delta}}{g \cdot g^{\sigma}}$

Example 2.8. To illustrate this generalization, we show two different instances. First let $\mathbb{T}=\mathbb{R}$. By the definition of the derivative and the fact that every $t \in \mathbb{T}^{\kappa} = \mathbb{T}$ is right-dense,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

If $\mathbb{T}=\mathbb{Z}$, by the definition we have for every $t \in \mathbb{T}^{\kappa} = \mathbb{T}$,

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{(t+1) - t} = f(t+1) - f(t) = \Delta f(t).$$

We note that throughout the dissertation, $f(\sigma(t))$ is denoted as $f^{\sigma}(t)$ and the notation that is used for an interval intersected with a time scale is $(a,b) \cap \mathbb{T} = (a,b)_{\mathbb{T}}$.

2.3 Integration

We now define functions that are integrable on arbitrary time scales $\mathbb{T}$.

Definition 2.6. We say a function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right- and left-sided limits exist at all right- and left-dense points in $\mathbb{T}$, respectively.

Definition 2.7. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (i.e. finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}).$$

It follows naturally that the set of functions $f : \mathbb{T} \to \mathbb{R}$ whose first n delta derivatives exist and are rd-continuous on $\mathbb{T}$ is denoted by

$$\mathrm{C}^{\mathrm{n}}_{\mathrm{rd}} = \mathrm{C}^{\mathrm{n}}_{\mathrm{rd}}(\mathbb{T}) = \mathrm{C}^{\mathrm{n}}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

From the previous two definitions we have the following theorem.

Theorem 2.2. *Assume* $f : \mathbb{T} \to \mathbb{R}$.

(i) *If f is continuous, then f is rd-continuous.*

(ii) *If f is rd-continuous, then f is regulated.*

(iii) *The forward jump operator σ is rd-continuous.*

(iv) *If f is regulated or rd-continuous, then so is f^σ.*

(v) *Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ is also regulated or rd-continuous, respectively.*

Definition 2.8. A continuous function $f : \mathbb{T} \to \mathbb{R}$ is *pre-differentiable* with (region of differentiation) D, provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.

The next theorem guarantees the existence of pre-antiderivatives.

Theorem 2.3. *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \qquad \textit{holds for all } t \in D.$$

Definition 2.9. Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 2.3 is called a *pre-antiderivative* of f.

Definition 2.10. The *indefinite integral* of a regulated function f is defined by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f.

Definition 2.11. The *Cauchy integral* is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

Definition 2.12. Suppose that $\sup(\mathbb{T}) = \infty$. The *improper integral* is defined by

$$\int_a^\infty f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t \quad \text{for all } a \in \mathbb{T}.$$

Definition 2.13. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^\Delta(t) = f(t) \quad \text{holds for all } t \in \mathbb{T}^\kappa.$$

2.4 Hilger's Complex Plane

Definition 2.14. For $h > 0$, we define the *Hilger complex numbers*, the *Hilger real axis*, the *Hilger alternating axis*, and the *Hilger imaginary circle* as

$$\begin{aligned}
\mathbb{C}_h &:= \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}, \\
\mathbb{R}_h &:= \left\{ z \in \mathbb{C} : z \in \mathbb{R} \text{ and } z > -\frac{1}{h} \right\}, \\
\mathbb{A}_h &:= \left\{ z \in \mathbb{C} : z \in \mathbb{R} \text{ and } z < -\frac{1}{h} \right\}, \\
\mathbb{I}_h &:= \left\{ z \in \mathbb{C} : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\},
\end{aligned}$$

respectively. For $h = 0$, let $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{A}_0 := \emptyset$, and $\mathbb{I}_0 := i\mathbb{R}$.

Definition 2.15. Let $h > 0$ and $z \in \mathbb{C}_h$. The *Hilger real part of z* is defined by

$$\mathrm{Re}_h(z) := \frac{|zh + 1| - 1}{h}$$

and the *Hilger imaginary part of z* is defined by

$$\mathrm{Im}_h(z) := \frac{\mathrm{Arg}(zh + 1)}{h},$$

where Arg(z) denotes the principal argument of z (i.e., $-\pi < \mathrm{Arg}(z) \leq \pi$).

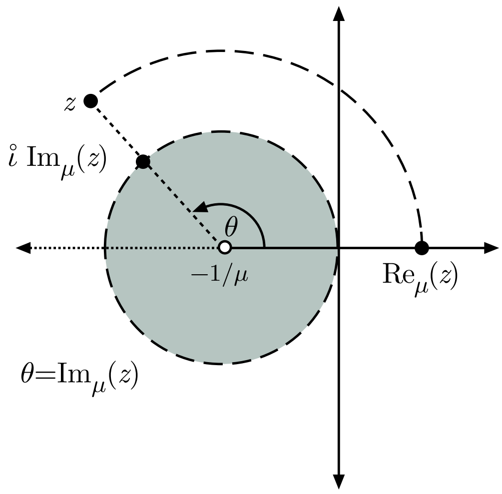


Figure 2.4. The Hilger complex plane.

2.5 The Regressive Group

Definition 2.16. The function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if

$$1 + \mu(t)p(t) \neq 0, \qquad t \in \mathbb{T}^\kappa.$$

From this point, all regressive *and* rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted as

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

Definition 2.17. The operation $\oplus$ (read "circle plus") on $\mathcal{R}$ is defined by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t), \quad \text{for all } t \in \mathbb{T}^\kappa, \quad p, q \in \mathcal{R}.$$

The following theorem will prove very useful throughout the paper and is stated here without proof [6].

Theorem 2.4. $(\mathcal{R}(\mathbb{T}, \mathbb{R}), \oplus)$ *is an Abelian group.*

From this point, we call $\mathcal{R} = (\mathcal{R}(\mathbb{T}, \mathbb{R}), \oplus)$ the *regressive group.*

Corollary 2.1. *The set of all positively regressive elements of $\mathcal{R}$ defined by*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \ \textit{for all } t \in \mathbb{T}^\kappa\}$$

is a subgroup of $\mathcal{R}$.

Definition 2.18. The function $p : \mathbb{T} \to \mathbb{R}$ is *uniformly regressive* on $\mathbb{T}$ if there exists a positive constant δ such that

$$0 < \delta^{-1} \leq |1 + \mu(t)p(t)|, \qquad t \in \mathbb{T}^\kappa.$$

Definition 2.19. The function $\ominus p$ is defined by

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)}, \qquad \text{for all } t \in \mathbb{T}^\kappa, \quad p \in \mathcal{R}.$$

Definition 2.20. The operation $\ominus$ (read "circle minus") on $\mathcal{R}$ is defined by

$$(p \ominus q)(t) = (p \oplus (\ominus q))(t), \qquad \text{for all } t \in \mathbb{T}^\kappa, \quad p, q \in \mathcal{R}.$$

We remark that if $p, q \in \mathcal{R}$, then $\ominus p$, $\ominus q$, $p \oplus q$, $p \ominus q$, $q \ominus p \in \mathcal{R}$.

2.6 The Time Scale Exponential Function

We employ a cylinder transform, defined below, to define the generalized time scale exponential function for an arbitrary time scale $\mathbb{T}$.

Definition 2.21. For $h > 0$, let $\mathbb{Z}_h$ be the strip

$$\mathbb{Z}_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h}\}$$

and for $h = 0$, let $\mathbb{Z}_0 := \mathbb{C}$.

Definition 2.22. For $h > 0$, the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined by

$$\xi_h(z) = \frac{1}{h} \text{Log}(1 + zh), \tag{2.1}$$

where Log is the principal logarithm function. Note that when $h = 0$, we define $\xi_0(z) = z$, for all $z \in \mathbb{C}$. The inverse cylinder transformation $\xi_h^{-1} : \mathbb{Z}_h \to \mathbb{C}_h$ is defined by

$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h}. \tag{2.2}$$

See Figure 2.5.

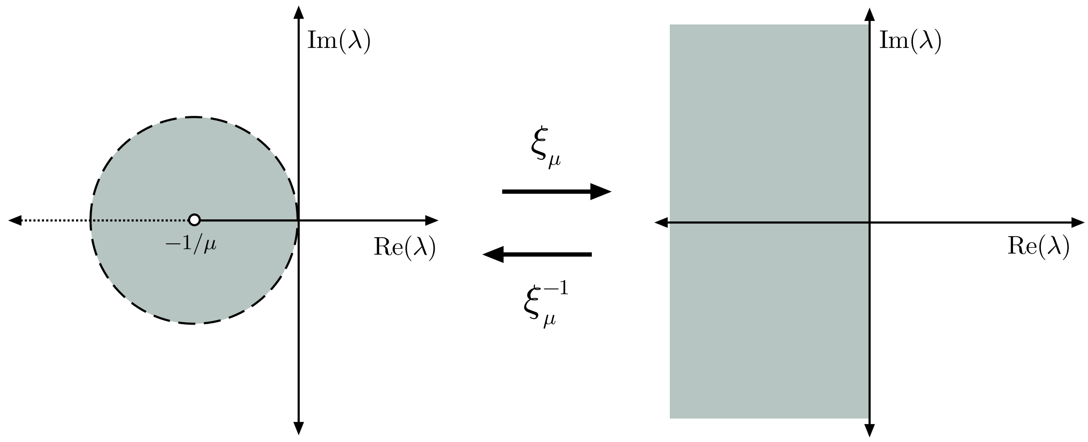


Figure 2.5: The cylinder (2.1) and inverse cylinder (2.2) transformations map the familiar stability region in the continuous case to the interior of the Hilger circle in the general time scale case.

Now we define the generalized time scale exponential function. We list some properties in the following lemma and refer the reader to [6] for a complete summary.

Definition 2.23. If $p \in \mathcal{R}$, then we define the *generalized time scale exponential function* by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \qquad \text{for all } s,t \in \mathbb{T}.$$

Lemma 2.1. [6] *Some properties of the generalized exponential are the following:*

(i) *If $p \in \mathcal{R}$, then semigroup property $e_p(t,r)e_p(r,s) = e_p(t,s)$ is satisfied for all $r,s,t \in \mathbb{T}$.*

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$.

(iii) *If $p \in \mathcal{R}^+$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.*

(iv) *If $1+\mu(t)p(t) < 0$ for some $t \in \mathbb{T}$, then $e_p(t,t_0)e_p(\sigma(t),t_0) < 0$.*

(v) *If $\mathbb{T} = \mathbb{R}$, then $e_p(t,s) = e^{\int_s^t p(\tau)d\tau}$. Moreover, if p is constant, then $e_p(t,s) = e^{p(t-s)}$.*

(vi) *If $\mathbb{T} = \mathbb{Z}$, then $e_p(t,s) = \prod_{\tau=s}^{t-1}(1+p(\tau))$. Moreover, if $\mathbb{T} = h\mathbb{Z}$, with $h > 0$ and p is constant, then $e_p(t,s) = (1+hp)^{\frac{(t-s)}{h}}$.*

Definition 2.24. If $p \in \mathcal{R}$ and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, then the dynamic equation

$$y^{\Delta}(t) = p(t)y(t) + f(t) \tag{2.3}$$

is called *regressive.*

Theorem 2.5 (Variation of Constants). *If* (2.3) *is regressive,* t_0 *is fixed in* $\mathbb{T}$ *and* $y(t_0) = y_0 \in \mathbb{R}$*, then the unique solution to the first order dynamic equation on* $\mathbb{T}$

$$y^{\Delta}(t) = p(t)y(t) + f(t), \qquad y(t_0) = y_0,$$

exists and is given by

$$y(t) = y_0 e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau)) f(\tau) \Delta\tau.$$

2.7 Regressive Matrices

We now introduce the concept of a regressive matrix, "circle plus" addition, "circle minus" substraction, and the time scale matrix exponential.

Definition 2.25. Let A be an $m \times n$-matrix-valued function on a time scale $\mathbb{T}$. We say that A is *rd-continuous* on $\mathbb{T}$ if each entry of A is rd-continuous, and the class of all such rd-continuous $m \times n$-matrix-valued functions on $\mathbb{T}$ is denoted by

$$\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^{m \times n}).$$

Definition 2.26. An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called *regressive* (with respect to $\mathbb{T}$) provided

$$I + \mu(t)A(t) \text{ is invertible for all } t \in \mathbb{T}^{\kappa},$$

and the class of all such regressive and rd-continuous functions is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}).$$

We say the $n \times 1$-vector-valued IVP

$$y^{\Delta}(t) = A(t)y(t) + f(t), \quad y(t_0) = y_0 \tag{2.4}$$

is *regressive* provided $A \in \mathcal{R}$ and $f : \mathbb{T} \to \mathbb{R}^n$ is a rd-continuous vector-valued function.

The next lemma provides a fact about the relationship between the $n \times n$-matrix-valued function A and the eigenvalues $\lambda_i(t)$ of $A(t)$.

Lemma 2.2. *The $n \times n$-matrix-valued function A is regressive if and only if the eigenvalues of $\lambda_i(t)$ of $A(t)$ are regressive for all $1 \leq i \leq n$.*

Definition 2.27. Assume that A and B are regressive $n \times n$-matrix-valued functions on $\mathbb{T}$. Then we define the following operations

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t),$$

$$(\ominus A)(t) = -\left[I + \mu(t)A(t)\right]^{-1} A(t) = -A(t)\left[I + \mu(t)A(t)\right]^{-1},$$

and

$$(A \ominus B)(t) = (A \oplus (\ominus B))(t),$$

for all $t \in \mathbb{T}^\kappa$.

Theorem 2.6. *$(\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n}), \oplus)$ is a group.*

From this theorem, we know that whenever A, $B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, then $A \oplus B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

We now state some properties of the regressive matrix-valued functions A and B. We let A^* denote the conjugate transpose of A. If $A \in \mathbb{R}^{m\times n}$, then $A^* = A^T$.

Lemma 2.3. *Suppose that A and B are regressive matrix-valued functions taking on complex values. Then we have the following:*

(i) *A^* is regressive;*

(ii) *$A^* \oplus B^* = (A \oplus B)^*$.*

Now the generalized matrix exponential function from [6] is presented.

Definition 2.28. Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}$ is an $n\times n$-matrix-valued function. The unique matrix-valued solution to the IVP

$$Y^{\Delta}(t) = A(t)Y(t), \qquad Y(t_0) = I_n, \tag{2.5}$$

where I_n is the $n \times n$-identity matrix, is called the *time scale matrix exponential function* (at t_0), and it is denoted by $e_A(t, t_0)$, where the subscript A may be a time varying or a constant matrix. It can also be called the *transition matrix* for the system (2.5).

In this dissertation, we denote the solution to (2.5) as $\Phi_A(t, t_0)$ when $A(t)$ is time varying and note that $\Phi_A(t, t_0) \equiv e_A(t, t_0)$ only when $A(t) \equiv A$ is a constant matrix. Also, if $A(t)$ is a function on $\mathbb{T}$ and the time scale matrix exponential function is a function on some other time scale $\mathbb{S}$, then $A(t)$ is constant with respect to $e_{A(t)}(\tau, s)$, for all $\tau, s \in \mathbb{S}$ and $t \in \mathbb{T}$. We state the following lemma which lists some properties of the transition matrix $\Phi_A(t, t_0)$ and a theorem that guarantees a unique solution to the regressive $n \times 1$-vector-valued dynamic IVP (2.4) that is used throughout.

Lemma 2.4. *Suppose $A \in \mathcal{R}$ are matrix-valued functions on $\mathbb{T}$. Then*

(i) *The semigroup property $\Phi_A(t, r)\Phi_A(r, s) = \Phi_A(t, s)$ is satisfied for all $r, s, t \in \mathbb{T}$.*

(ii) $\Phi_A(\sigma(t), s) = (I + \mu(t)A(t))\Phi_A(t, s)$.

(iii) *If $\mathbb{T} = \mathbb{R}$ and A is constant, then $\Phi_A(t, s) = e_A(t, s) = e^{A(t-s)}$.*

(iv) *If $\mathbb{T} = h\mathbb{Z}$, with $h > 0$, and A is constant, then $\Phi_A(t, s) = e_A(t, s) = (I + hA)^{\frac{(t-s)}{h}}$.*

Theorem 2.7 (Variation of Constants). *Let $t_0 \in \mathbb{T}$ and $y(t_0) = y_0 \in \mathbb{R}^n$. Then the regressive IVP (2.4) has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$ given by*

$$y(t) = \Phi_A(t, t_0)y_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

CHAPTER THREE

General Definitions and Preliminary Stability Results

3.1 Matrix Norms and Definiteness

We start by introducing some notation that will be employed in the sequel.

Definition 3.1. The *Euclidean norm* of an $n \times 1$ vector $x(t)$ is defined to be a real-valued function of t and is denoted by

$$||x(t)|| = \sqrt{x^T(t)x(t)}.$$

Definition 3.2. The *induced norm* of an $m \times n$ matrix A is defined to be

$$||A|| = \max_{||x||=1} ||Ax||.$$

We remark that the norm of A induced by the Euclidean norm above is equal to the nonnegative square root of the absolute value of the largest eigenvalue of the symmetric matrix $A^T A$. Thus, we define this norm next.

Definition 3.3. The *spectral norm* of an $m \times n$ matrix A is defined to be

$$||A|| = \left[\max_{||x||=1} x^T A^T A x\right]^{\frac{1}{2}}.$$

This will be the matrix norm that is used in the sequel and will be denoted by $|| \cdot ||$.

Definition 3.4. A symmetric matrix M is defined to be *positive semidefinite* if for all $n \times 1$ vectors x

$$x^T M x \geq 0$$

and is *positive definite* if

$$x^T M x \geq 0, \quad \text{with equality only when } x = 0.$$

Negative semidefiniteness and definiteness are defined in terms of positive semidefiniteness and definiteness of $-M$.

3.2 Stability Definitions

We now define the concepts of uniform stability and uniform exponential stability. These two concepts involve the boundedness of the solutions of the regressive time varying linear dynamic equation

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0, \qquad t_0 \in \mathbb{T}. \tag{3.1}$$

Definition 3.5. The time varying linear dynamic equation (3.1) is *uniformly stable* if there exists a finite constant $\gamma > 0$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$||x(t)|| \leq \gamma ||x(t_0)||, \qquad t \geq t_0.$$

For the next definition, we define a stability property that not only concerns the boundedness of a solutions to (3.1), but also the asymptotic characteristics of the solutions as well. If the solutions to (3.1) possess the following stability property, then the solutions approach zero exponentially as $t \to \infty$ (i.e. the norms of the solutions are bounded above by a decaying exponential function).

Definition 3.6. The time varying linear dynamic equation (3.1) is called *uniformly exponentially stable* if there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$||x(t)|| \leq ||x(t_0)|| \gamma e_{-\lambda}(t, t_0), \qquad t \geq t_0.$$

It is obvious by inspection of the previous definitions that we must have $\gamma \geq 1$. By using the word uniform, it is implied that the choice of γ does not depend on the initial time t_0.

The last stability definition given uses a uniformity condition to conclude exponential stability.

Definition 3.7. The linear state equation (3.1) is defined to be *uniformly asymptotically stable* if it is uniformly stable and given any $\delta > 0$, there exists a $T > 0$ so that for any t_0 and $x(t_0)$, the corresponding solution $x(t)$ satisfies

$$||x(t)|| \leq \delta ||x(t_0)||, \quad t \geq t_0 + T. \tag{3.2}$$

It is noted that the time T that must pass before the norm of the solution satisfies (3.2) and the constant $\delta > 0$ is independent of the initial time t_0.

3.3 Stability Characterizations

We now state and prove four theorems, the first three of which characterize uniform stability and uniform exponential stability in terms of the transition matrix for the system (3.1). The forth theorem illustrates the relationship between uniform asymptotic stability and uniform exponential stability.

Theorem 3.1. *The time varying linear dynamic equation* (3.1) *is uniformly stable if and only if there exists a* $\gamma > 0$ *such that*

$$||\Phi_A(t, t_0)|| \leq \gamma$$

for all $t \geq t_0$ *with* $t,\ t_0 \in \mathbb{T}$.

Proof. Suppose that (3.1) is uniformly stable. Then there exists a $\gamma > 0$ such that for any $t_0,\ x(t_0)$, the solutions satisfy

$$||x(t)|| \leq \gamma ||x(t_0)||, \quad t \geq t_0.$$

Given any t_0 and $t_a \geq t_0$, let x_a be a vector such that

$$||x_a|| = 1, \quad ||\Phi_A(t_a, t_0)x_a|| = ||\Phi_A(t_a, t_0)||\ ||x_a|| = ||\Phi_A(t_a, t_0)||$$

So the initial state $x(t_0) = x_a$ gives a solution of (3.1) that at time t_a satisfies

$$||x(t_a)|| = ||\Phi_A(t_a, t_0)x_a|| = ||\Phi_A(t_a, t_0)||\ ||x_a|| \leq \gamma ||x_a||.$$

Since $||x_a|| = 1$, we see that $||\Phi_A(t_a, t_0)|| \leq \gamma$. Since x_a can be selected for any t_0 and $t_a \geq t_0$, we see that $||\Phi_A(t, t_0)|| \leq \gamma$ for all $t,\ t_0 \in \mathbb{T}$.

Now suppose that there exists a γ such that $||\Phi_A(t, t_0)|| \leq \gamma$ for all $t,\ t_0 \in \mathbb{T}$. For any t_0 and $x(t_0) = x_0$, the solution of (3.1) satisfies

$$||x(t)|| = ||\Phi_A(t, t_0)x_0|| \leq ||\Phi_A(t, t_0)||\, ||x_0|| \leq \gamma ||x_0||, \quad t \geq t_0.$$

Thus, uniform stability of (3.1) is established. □

Theorem 3.2. *The time varying linear dynamic equation* (3.1) *is uniformly exponentially stable if and only if there exist* $\lambda,\ \gamma > 0$ *with* $-\lambda \in \mathcal{R}^+$ *such that*

$$||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$$

for all $t \geq t_0$ *with* $t,\ t_0 \in \mathbb{T}$.

Proof. First suppose that (3.1) is exponentially stable. Then there exist $\gamma,\ \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that for any t_0 and $x_0 = x(t_0)$, the solution of (3.1) satisfies

$$||x(t)|| = ||x_0||\gamma e_{-\lambda}(t, t_0), \quad t \geq t_0.$$

So for any t_0 and $t_a \geq t_0$, let x_a be a vector such that

$$||x_a|| = 1, \quad ||\Phi_A(t_a, t_0)x_a|| = ||\Phi_A(t_a, t_0)||\, ||x_a|| = ||\Phi_A(t_a, t_0)||.$$

Then the initial state $x(t_0) = x_a$ gives a solution of (3.1) that at time t_a satisfies

$$||x(t_a)|| = ||\Phi_A(t_a, t_0)x_a|| = ||\Phi_A(t_a, t_0)||\, ||x_a|| \leq ||x_a||\gamma e_{-\lambda}(t, t_0).$$

Since $||x_a|| = 1$ and $-\lambda \in \mathcal{R}^+$, we have $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$. Since x_a can be selected for any t_0 and $t_a \geq t_0$, we see that $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ for all $t,\ t_0 \in \mathbb{T}$.

Now suppose there exist $\gamma,\ \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ for all $t,\ t_0 \in \mathbb{T}$. For any t_0 and $x(t_0) = x_0$, the solution of (3.1) satisfies

$$||x(t)|| \leq ||\Phi_A(t, t_0)x_0|| \leq ||\Phi_A(t, t_0)||\, ||x_0|| \leq ||x_0||\gamma e_{-\lambda}(t, t_0), \quad t \geq t_0,$$

and thus uniform exponential stability is attained. □

Theorem 3.3. *Suppose there exists a constant α such that for all $t \in \mathbb{T}$, $||A(t)|| \leq \alpha$. Then the linear state equation* (3.1) *is uniformly exponentially stable if and only if there exists a constant β such that*

$$\int_{\tau}^{t} ||\Phi_A(t, \sigma(s))||\Delta s \leq \beta \tag{3.3}$$

for all $t, \tau \in \mathbb{T}$ with $t \geq \sigma(\tau)$.

Proof. Suppose that the state equation (3.1) is uniformly exponentially stable. By Theorem 3.2, there exist $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ so that

$$||\Phi_A(t, \tau)|| \leq \gamma e_{-\lambda}(t, \tau)$$

for all $t, \tau \in \mathbb{T}$ with $t \geq \tau$. So we now see that by a result in [6, Thm. 2.39],

$$\begin{aligned} \int_{\tau}^{t} ||\Phi_A(t, \sigma(s))||\Delta s &\leq \int_{\tau}^{t} \gamma e_{-\lambda}(t, \sigma(s))\Delta s \\ &= \frac{\gamma}{\lambda}\left[e_{-\lambda}(t, t) - e_{-\lambda}(t, \tau)\right] \\ &= \frac{\gamma}{\lambda}\left[1 - e_{-\lambda}(t, \tau)\right] \\ &\leq \frac{\gamma}{\lambda}, \end{aligned}$$

for all $t \geq \sigma(\tau)$. Thus, we have established (3.3) with $\beta = \frac{\gamma}{\lambda}$.

Now suppose that (3.3) holds. We see that we can represent the state transition matrix as

$$\Phi_A(t, \tau) = I - \int_{\tau}^{t} [\Phi_A(t, s)]^{\Delta s}\Delta s = I + \int_{\tau}^{t} \Phi_A(t, \sigma(s))A(s)\Delta s,$$

so that, with $||A(t)|| \leq \alpha$,

$$||\Phi_A(t, \tau)|| \leq 1 + \int_{\tau}^{t} ||\Phi_A(t, \sigma(s))|| \, ||A(s)||\Delta s \leq 1 + \alpha\beta,$$

for all $t, \tau \in \mathbb{T}$ with $t \geq \sigma(\tau)$.

To complete the proof,

$$\begin{aligned}
||\Phi_A(t,\tau)||(t-\tau) &= \int_\tau^t ||\Phi_A(t,\tau)||\Delta s \\
&\leq \int_\tau^t ||\Phi_A(t,\sigma(s))||\,||\Phi_A(\sigma(s),\tau)||\Delta s \\
&\leq \beta(1+\alpha\beta), \qquad (3.4)
\end{aligned}$$

for all $t \geq \sigma(\tau)$.

Now, choosing T with $T \geq 2\beta(1+\alpha\beta)$ and $t = \tau + T \in \mathbb{T}$, we obtain

$$||\Phi_A(t,\tau)|| \leq \frac{1}{2}, \quad t,\tau \in \mathbb{T}. \qquad (3.5)$$

Using the bound from equation (3.4) and (3.5), we have the following set of inequalities on intervals in the time scale of the form $[\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$, with arbitrary τ:

$$\begin{aligned}
||\Phi_A(t,\tau)|| &\leq 1+\alpha\beta, \quad t \in [\tau, \tau+T)_{\mathbb{T}} \\
||\Phi_A(t,\tau)|| &= ||\Phi_A(t,\tau+T)\Phi_A(\tau+T,\tau)|| \\
&\leq ||\Phi_A(t,\tau+T)||\,||\Phi_A(\tau+T,\tau)|| \\
&\leq \frac{1+\alpha\beta}{2}, \quad t \in [\tau+T, \tau+2T)_{\mathbb{T}} \\
||\Phi_A(t,\tau)|| &= ||\Phi_A(t,\tau+2T)\Phi_A(\tau+2T,\tau+T)\Phi_A(\tau+T,\tau)|| \\
&\leq ||\Phi_A(t,\tau+2T)||\,||\Phi_A(\tau+2T,\tau+T)||\,||\Phi_A(\tau+T,\tau)|| \\
&\leq \frac{1+\alpha\beta}{2^2}, \quad t \in [\tau+2T, \tau+3T)_{\mathbb{T}}\,.
\end{aligned}$$

In general, for any $\tau \in \mathbb{T}$, we have

$$||\Phi_A(t,\tau)|| \leq \frac{1+\alpha\beta}{2^k}, \quad t \in [\tau+kT, \tau+(k+1)T)_{\mathbb{T}}\,.$$

We now choose the bounds to obtain a decaying exponential bound. Let $\gamma = 2(1+\alpha\beta)$ and define the positive (possibly piecewise defined) function $\lambda(t)$ (with

$-\lambda(t) \in \mathcal{R}^+)$ as the solution to $e_{-\lambda}(t,\tau) \geq e_{-\lambda}(\tau + (k+1)T, \tau) = \frac{1}{2^{k+1}}$, for $t \in [\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$ with $k \in \mathbb{N}_0$. Then for all $t, \tau \in \mathbb{T}$ with $t \geq \tau$, we obtain the decaying exponential bound

$$||\Phi_A(t,\tau)|| \leq \gamma e_{-\lambda}(t,\tau).$$

Therefore, by Theorem 3.2, we have uniform exponential stability. □

For example, when $\mathbb{T} = \mathbb{R}$, the solution to

$$e^{-\lambda(t-\tau)} \geq e^{-\lambda(\tau+(k+1)T-\tau)} = e^{-\lambda((k+1)T)} = \frac{1}{2^{k+1}}$$

with $k \in \mathbb{N}_0$ and $t \in [\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$ is $\lambda = -\frac{1}{T}\ln(\frac{1}{2})$.

When $\mathbb{T} = \mathbb{Z}$, the solution to

$$(1-\lambda)^{t-\tau} \geq (1-\lambda)^{\tau+(k+1)T-\tau} = (1-\lambda)^{(k+1)T} = \frac{1}{2^{k+1}}$$

with $k \in \mathbb{N}_0$ and $t \in [\tau + kT, \tau + (k+1)T)_{\mathbb{T}}$ is $\lambda = 1 - \left(\frac{1}{2}\right)^{-\frac{1}{T}}$, and $-\lambda \in \mathcal{R}^+$.

Theorem 3.4. *The linear state equation* (3.1) *is uniformly exponentially stable if and only if it is uniformly asymptotically stable.*

Proof. Suppose that the system (3.1) is uniformly exponentially stable. This implies that there exist constants $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$ so that $||\Phi_A(t,\tau)|| \leq \gamma e_{-\lambda}(t,\tau)$ for $t \geq \tau$. Clearly, this implies uniform stability. Now, given a $\delta > 0$, we choose a sufficiently large positive constant $T > 0$ such that $t_0 + T \in \mathbb{T}$ and $e_{-\lambda}(t_0 + T, t_0) \leq \frac{\delta}{\gamma}$. Then for any t_0 and x_0, and $t \geq T + t_0$ with $t \in \mathbb{T}$,

$$\begin{aligned}
||x(t)|| &= ||\Phi_A(t,t_0)x_0|| \\
&\leq ||\Phi_A(t,t_0)||\;||x_0|| \\
&\leq \gamma e_{-\lambda}(t,t_0)||x_0|| \\
&\leq \gamma e_{-\lambda}(t_0+T,t_0)||x_0|| \\
&\leq \delta||x_0||, \quad t \geq t_0 + T.
\end{aligned}$$

Thus, (3.1) is uniformly asymptotically stable.

Now suppose the converse. By definition of uniform asymptotic stability, (3.1) is uniformly stable. Thus, there exists a constant $\gamma > 0$ so that

$$||\Phi_A(t,\tau)|| \leq \gamma, \qquad \text{for all } t \geq \tau. \tag{3.6}$$

Choosing $\delta = \frac{1}{2}$, let T be a positive constant so that $t = t_0 + T \in \mathbb{T}$ and (3.2) is satisfied. Given a t_0 and letting x_a be so that $||x_a|| = 1$, we have

$$||\Phi_A(t_0+T,t_0)x_a|| = ||\Phi_A(t_0+T,t_0)||.$$

When $x_0 = x_a$, the solution $x(t)$ of (3.1) satisfies

$$||x(t)|| = ||x(t_0+T)|| = ||\Phi_A(t_0+T,t_0)x_a|| = ||\Phi_A(t_0+T,t_0)||\,||x_a|| \leq \frac{1}{2}||x_a||.$$

From this, we obtain

$$||\Phi_A(t_0+T,t_0)|| \leq \frac{1}{2}. \tag{3.7}$$

It can be seen that for any t_0 there exists an x_a as claimed. Therefore, the above inequality holds for any t_0. Thus, by using (3.6) and (3.7) exactly as in Theorem 3.3, uniform exponential stability is obtained. $\square$

CHAPTER FOUR

Lyapunov Stability Criteria for Linear Dynamic Systems

4.1 Stability of the Time Varying Linear Dynamic System

In this section, we investigate the stability of the regressive time varying linear dynamic system of the form

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0, \qquad t_0 \in \mathbb{T}. \tag{4.1}$$

We are seeking to assess the stability of the unforced, dissipative system by observing the system's total energy as the state of the system evolves in time. If the total energy of the system decreases as the state evolves, then the state vector approaches a constant value (equilibrium point) corresponding to zero energy as time increases. The stability of the system involves the growth characteristics of solutions of the state equation (4.1), and these properties can be measured by a suitable (energy-like) scalar function of the state vector. In the following two subsections, we discuss the boundedness properties and asymptotic behavior as $t \to \infty$ of solutions of the system (4.1). Of course, the problem at hand is obtaining a proper scalar function.

We assume that our time scale $\mathbb{T}$ is unbounded above. To start, we consider conditions that imply all solutions of the linear state equation (4.1) are such that $||x(t)||^2 \to 0$ as $t \to \infty$. For any solution of (4.1), the delta derivative of the scalar function

$$||x(t)||^2 = x^T(t)x(t)$$

with respect to t is:

$$\begin{aligned}\left[||x(t)||^2\right]^{\Delta_t} &= x^{T^{\Delta}}(t)x(t) + x^{T^{\sigma}}(t)x^{\Delta}(t)\\ &= x^T(t)A^T(t)x(t) + x^T(t)(I + \mu(t)A^T(t))A(t)x(t)\\ &= x^T(t)[A^T(t) + A(t) + \mu(t)A^T(t)A(t)]x(t).\end{aligned}$$

So if the quadratic form we obtain is negative definite, i.e. $A^T(t)+A(t)+\mu(t)A^T(t)A(t)$ is negative definite at each t, then $||x(t)||^2$ will decrease monotonically as t increases. We later show that if there exists a $\nu > 0$ so that $A^T(t)+A(t)+\mu(t)A^T(t)A(t) \leq -\nu I$ for all t, then $||x(t)||^2 \to 0$ as $t \to \infty$. To formalize our discussion, we define time-dependent quadratic forms that are useful for analyzing stability. We will refer to these quadratic forms as *unified time scale quadratic Lyapunov functions.* For a symmetric matrix $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ we write the general quadratic Lyapunov function as $x^T(t)Q(t)x(t)$. If $x(t)$ is a solution to (4.1), then interest lies in the behavior of the (scalar) quantity $x^T(t)Q(t)x(t)$ for $t \geq t_0$. With this we now define one of the main ideas of this dissertation.

Definition 4.1. Let $Q(t)$ be a symmetric matrix such that $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$. A *unified time scale quadratic Lyapunov function* is given by

$$x^T(t)Q(t)x(t), \qquad t \geq t_0, \tag{4.2}$$

with delta derivative

$$\begin{aligned}
[x^T(t)Q(t)x(t)]^{\Delta_t} &= x^T(t)[A^T(t)Q(t) \\
&\quad + (I+\mu(t)A^T(t))(Q^\Delta(t)+Q(t)A(t)+\mu(t)Q^\Delta(t)A(t))]x(t) \\
&= x^T(t)[A^T(t)Q(t)+Q(t)A(t)+\mu(t)A^T(t)Q(t)A(t) \\
&\quad + (I+\mu(t)A^T(t))Q^\Delta(t)(I+\mu(t)A(t))]x(t).
\end{aligned}$$

The matrix dynamic equation that is obtained by differentiating (4.2) with respect to t is given by

$$\begin{aligned}
&A^T(t)Q(t)+Q(t)A(t)+\mu(t)A^T(t)Q(t)A(t) \\
&\qquad + (I+\mu(t)A^T(t))Q^\Delta(t)(I+\mu(t)A(t)) = -M, \qquad M = M^T.
\end{aligned} \tag{4.3}$$

One can see that it merges with the familiar continuous matrix differential equation ($\mathbb{T} = \mathbb{R}$) and discrete ($\mathbb{T} = \mathbb{Z}$) difference (recursive) equation obtained from the respective quadratic Lyapunov functions in $\mathbb{R}$ and $\mathbb{Z}$.

For the continuous case $\mathbb{T} = \mathbb{R}$, we observe that $\mu(t) \equiv 0$. Thus, from (4.1) we now have the continuous system

$$\dot{x}(t) = A(t)x(t), \qquad x(t_0) = x_0. \tag{4.4}$$

The quadratic Lyapunov function that emerges from (4.4) is

$$\frac{d}{dt}[x^T(t)Q(t)x(t)] = x^T(t)[A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t)]x(t),$$

where

$$A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) = -M, \qquad M = M^T,$$

is the familiar matrix differential equation [7, 13, 30, 41, 42] derived from the continuous system (4.4).

For the discrete case $\mathbb{T} = \mathbb{Z}$, we note that systems of difference equations in $\mathbb{Z}$ are traditionally written in recursive form

$$x(t+1) = A_R(t)x(t), \qquad x(t_0) = x_0, \tag{4.5}$$

while the difference form is written

$$x^\Delta(t) = \Delta x(t) = x(t+1) - x(t) = A(t)x(t), \qquad x(t_0) = x_0. \tag{4.6}$$

Thus, changing from difference form to recursion just requires a unit shift on the matrix $A(t)$, that is,

$$x(t+1) = (I + A(t))x(t) = A_R(t)x(t)$$

where $A_R = (I + A)$.

Now taking the forward difference of the unified time scale quadratic Lyapunov function (4.2) with respect to (4.6), and noting that $\mu(t) \equiv 1$ when in $\mathbb{Z}$, we obtain

$$
\begin{aligned}
&x^T(t)[A^T(t)Q(t) + Q(t)A(t) + A^T(t)Q(t)A(t) \\
&\qquad + (I + A^T(t))\Delta Q(t)(I + A(t))]x(t) \\
&= x^T(t)[(A_R^T(t) - I)Q(t) + Q(t)(A_R(t) - I) + (A_R^T(t) - I)Q(t)(A_R(t) - I) \\
&\qquad + A_R^T(t)\Delta Q(t)A_R(t)]x(t) \\
&= x^T(t)[A_R^T(t)Q(t) - Q(t) + Q(t)A_R(t) - Q(t) + A_R^T(t)Q(t)A_R(t) \\
&\qquad - A_R^T(t)Q(t) - Q(t)A_R(t) + Q(t) + A_R^T(t)\Delta Q(t)A_R(t)]x(t) \\
&= x^T(t)[-Q(t) + A_R^T(t)Q(t)A_R(t) + A_R^T(t)\Delta Q(t)A_R(t)]x(t) \\
&= x^T(t)[-Q(t) + A_R^T(t)Q(t)A_R(t) + A_R^T(t)(Q(t+1) - Q(t))A_R(t)]x(t) \\
&= x^T(t)[A_R^T(t)Q(t+1)A_R(t) - Q(t)]x(t),
\end{aligned}
$$

where

$$A_R^T(t)Q(t+1)A_R(t) - Q(t) = -M, \qquad M = M^T,$$

is the well known discrete matrix recursion equation [14, 31, 42] for the recursive system (4.5).

This shows that the unified time scale matrix dynamic equation merges into the continuous and discrete cases easily because of the time-varying graininess $\mu(t)$. This unified time scale matrix dynamic equation not only unifies the two special cases of continuous and discrete time, it also extends these notions to arbitrary time scales, and as such plays a crucial role in our analysis.

4.2 Uniform Stability

In this section, we introduce criteria for uniform stability of the system (4.1). The criteria introduced in Theorem 4.1 is a generalization of the Lyapunov criteria for uniform stability of discrete and continuous linear systems that can be found in the famous papers by Kalman and Bertram [30, 31]. Uniform stability involves the boundedness of all solutions of the system (4.1) and in the following theorem we derive sufficient conditions for uniform stability of the system. The strategy is to state requirements on the matrix $Q(t)$ so that the corresponding quadratic form yields uniform stability of the system.

Theorem 4.1. *The time varying linear dynamic system* (4.1) *is uniformly stable if for all* $t \in \mathbb{T}$, *there exists a symmetric matrix* $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *such that*

(i) $\eta I \leq Q(t) \leq \rho I$

(ii) $A^T(t)Q(t) + (I + \mu(t)A^T(t))(Q^\Delta(t) + Q(t)A(t) + \mu(t)Q^\Delta(t)A(t)) \leq 0,$

where $\eta, \rho \in \mathbb{R}^+$.

Proof. For any t_0 and $x(t_0) = x_0$, by (ii),

$$x^T(t)Q(t)x(t) - x^T(t_0)Q(t_0)x(t_0) = \int_{t_0}^{t} [x^T(s)Q(s)x(s)]^{\Delta_s}\Delta s \leq 0,$$

for $t \geq t_0$. Using (i),

$$\eta||x(t)||^2 \leq x^T(t)Q(t)x(t) \leq x^T(t_0)Q(t_0)x(t_0) \leq \rho||x(t_0)||^2,$$

which implies

$$||x(t)|| \leq \sqrt{\frac{\rho}{\eta}}||x(t_0)||.$$

Since this last statement holds for all t_0 and $x(t_0) = x_0$, equation (4.1) is uniformly stable. $\square$

To illustrate this theorem, we present an example.

Example 4.1. Consider the time varying linear dynamic system

$$x^\Delta(t) = \begin{bmatrix} -2 & 1 \\ -1 & -a(t) \end{bmatrix} x(t), \qquad x(t_0) = x_0,$$

where $a(t) \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ for all $t \in \mathbb{T}$. Choose $Q(t) = I$, so that $x^T(t)Q(t)x(t) = x^T(t)x(t) = ||x(t)||^2$. In Theorem 4.1, (i) is satisfied when $\eta = \rho = 1$. To satisfy the second requirement, we see for $Q(t) = I$, $Q^\Delta(t) = 0$ so

$$A^T(t)Q(t) + (I + \mu(t)A^T(t))(Q^\Delta(t) + Q(t)A(t) + \mu(t)Q^\Delta(t)A(t)) \leq 0$$

becomes

$$A^T(t) + A(t) + \mu(t)A^T(t)A(t) \leq 0.$$

Now

$$A(t) = \begin{bmatrix} -2 & 1 \\ -1 & -a(t) \end{bmatrix}, \quad A^T(t) = \begin{bmatrix} -2 & -1 \\ 1 & -a(t) \end{bmatrix},$$

and

$$\mu(t)A^T(t)A(t) = \mu(t) \begin{bmatrix} 5 & a(t)-2 \\ a(t)-2 & a(t)^2+1 \end{bmatrix},$$

so

$$A^T(t) + A(t) + \mu(t)A^T(t)A(t) = \begin{bmatrix} 5\mu(t)-4 & (a(t)-2)\mu(t) \\ (a(t)-2)\mu(t) & (a(t)^2+1)\mu(t)-2a(t) \end{bmatrix}.$$

For any 2×2 matrix

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

to be negative semidefinite, we need $-m_{11}, -m_{22} \geq 0$ and $\det(M) \geq 0$. For our matrix

$$A^*(t) := A^T(t) + A(t) + \mu(t)A^T(t)A(t),$$

we need

$$\begin{aligned} -a^*_{11} &= 4 - 5\mu(t) \geq 0 \text{ which implies } 0 \leq \mu(t) \leq \frac{4}{5}, \\ -a^*_{22} &= -((a(t)^2+1)\mu(t) - 2a(t)) \geq 0, \end{aligned}$$

and

$$\begin{aligned} \det(A^*(t)) = {} & 4\mu(t)^2a(t)^2 - 4\mu(t)a(t)^2 + 4\mu(t)^2a(t) \\ & - 10\mu(t)a(t) + 8a(t) + \mu(t)^2 - 4\mu(t) \geq 0. \end{aligned}$$

It is can be confirmed that for each $0 \leq \mu(t) \leq \frac{4}{5}$, the interval in which $-a^*_{22} \geq 0$ always contains the interval in which $\det(A^*(t)) \geq 0$. Thus, we only need to concern ourselves with the latter inequality. If $\mu(t) = \frac{4}{5}$, the only possible value that the function $a(t)$ may be is 2. If we let $\mu(t) = \frac{1}{2}$, we see that a window emerges for the allowable values of the function $a(t)$: $\frac{1}{2} \leq a(t) \leq \frac{7}{2}$. Letting $\mu(t) = \frac{2}{5}$, we see that another window develops for the allowable values of the function $a(t)$: $\frac{1}{3} \leq a(t) \leq \frac{9}{2}$. It is quite interesting to note that as $\mu(t) \to 0$, the window opens up to infinite length, bounded below by 0. Therefore, when $\mathbb{T} = \mathbb{R}$, the only requirement for $a(t)$ is that it is nonnegative for all $t \in \mathbb{T}$.

4.3 Uniform Exponential Stability

We now introduce sufficient criteria for uniform exponential stability of the system (4.1). The criteria introduced in Theorem 4.2 is again a generalization of the Lyapunov criteria for uniform exponential stability of discrete and continuous linear systems, which can be found in the companion papers of Kalman and Bertram [30, 31], as well as the classic text by Hahn [22]. There is a slight, but very powerful variation from uniform stability to uniform exponential stability. By requiring $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ to be symmetric, positive definite, and bounded above and below by positive definite matrices, along with a strictly negative definite delta derivative, i.e.

$$\left[x^T(t)Q(t)x(t)\right]^\Delta \leq -\epsilon\, x^T(t)x(t),$$

for some $\epsilon > 0$, we will show that all solutions of (4.1) are bounded above by a decaying exponential and go to zero as $t \to \infty$. Uniform exponential stability does imply that the system (4.1) is uniformly stable, but the converse is not true.

Theorem 4.2. *The time varying linear dynamic system* (4.1) *is uniformly exponentially stable if there exists a symmetric matrix* $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *such that for all* $t \in \mathbb{T}$

(i) $\eta I \leq Q(t) \leq \rho I$

(ii) $A^T(t)Q(t) + (I + \mu(t)A^T(t))(Q^\Delta(t) + Q(t)A(t) + \mu(t)Q^\Delta(t)A(t)) \leq -\nu I,$

where $\eta, \rho, \nu \in \mathbb{R}^+$ *and* $\frac{-\nu}{\rho} \in \mathcal{R}^+$.

Proof. For any initial condition t_0 and $x(t_0) = x_0$ with corresponding solution $x(t)$ of (4.1), we see that for all $t \geq t_0$, (ii) yields

$$[x^T(t)Q(t)x(t)]^\Delta \leq -\nu||x(t)||^2.$$

Also, for all $t \geq t_0$, (i) implies

$$x^T(t)Q(t)x(t) \leq \rho||x(t)||^2.$$

Thus

$$[x^T(t)Q(t)x(t)]^\Delta \leq \frac{-\nu}{\rho}x^T(t)Q(t)x(t),$$

for all $t \geq t_0$. Since $\frac{-\nu}{\rho} \in \mathcal{R}^+$, we can employ the time scale version of Gronwall's inequality [6] to obtain

$$x^T(t)Q(t)x(t) \leq x^T(t_0)Q(t_0)x(t_0)e_{\frac{-\nu}{\rho}}(t, t_0), \quad t \geq t_0. \tag{4.7}$$

By (i), $\eta I \leq Q(t)$ which is equivalent to $\eta||x(t)||^2 \leq x^T(t)Q(t)x(t)$ and division by η along with (4.7) yields

$$||x(t)||^2 \leq \frac{1}{\eta}x^T(t)Q(t)x(t) \leq \frac{1}{\eta}x^T(t_0)Q(t_0)x(t_0)e_{\frac{-\nu}{\rho}}(t, t_0), \quad t \geq t_0.$$

Since $x^T(t_0)Q(t_0)x(t_0) \leq \rho||x(t_0)||^2$, this implies

$$||x(t)||^2 \leq \frac{\rho}{\eta}||x(t_0)||^2 e_{\frac{-\nu}{\rho}}(t, t_0),$$

which yields

$$||x(t)|| \leq ||x(t_0)||\sqrt{\frac{\rho}{\eta}e_{\frac{-\nu}{\rho}}(t, t_0)}, \quad t \geq t_0.$$

This holds for arbitrary t_0 and $x(t_0)$. Thus, uniform exponential stability is obtained. $\square$

We present another example to show the difference between uniform stability and uniform exponential stability.

Example 4.2. Consider again the time varying linear dynamic system

$$x^{\Delta}(t) = \begin{bmatrix} -2 & 1 \\ -1 & -a(t) \end{bmatrix} x(t), \qquad x(t_0) = x_0,$$

where we now let $a(t) = \sin(t) + 2$ which is obviously in $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ for all $t \in \mathbb{T}$. We note that $\sin(t)$ is the usual sine function that gives the sine value of each point in $\mathbb{T}$ and it is *not* the time scale function $\sin_1(t, 0)$.

Again, choose $Q(t) = I$, so that $x^T(t)Q(t)x(t) = x^T(t)x(t) = ||x(t)||^2$. In Theorem 4.1, (i) is satisfied when $\eta = \rho = 1$. To satisfy the second requirement, we see $Q(t) = I$, so $Q^{\Delta}(t) = 0$ and thus

$$A^T(t)Q(t) + (I + \mu(t)A^T(t))(Q^{\Delta}(t) + Q(t)A(t) + \mu(t)Q^{\Delta}(t)A(t)) \leq -\nu I$$

becomes

$$A^T(t) + A(t) + \mu(t)A^T(t)A(t) \leq -\nu I.$$

For any 2×2 matrix

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

to be negative definite, we need $-m_{11} > 0$ and $\det(M) > 0$. For our matrix

$$A^*(t) := A^T(t) + A(t) + \mu(t)A^T(t)A(t),$$

we need $-a^*_{11} = 4 - 5\mu(t) > 0$ which implies that $0 \leq \mu(t) < \frac{4}{5}$ and

$$\begin{aligned} \det(A^*(t)) = {} & 4\sin^2(t)\mu(t)^2 + 20\sin(t)\mu(t)^2 + 25\mu(t)^2 \\ & - 4\sin^2(t)\mu(t) - 26\sin(t)\mu(t) - 40\mu(t) + 8\sin(t) + 16 > 0. \end{aligned}$$

We note that $\det(A^*(t)) > 0$ for all $t \in \mathbb{T}$ as long as $0 \leq \mu(t) < \frac{1}{2}$.

For instance, letting $\mathbb{T} = \mathbb{P}_{.6,.4} = \bigcup_{k=0}^{\infty} [k, k + .6]$, in this time scale

$$\mu(t) = \begin{cases} 0, & \text{if } t \in \bigcup_{k=0}^{\infty} [k, k + .6), \\ .4, & \text{if } t \in \bigcup_{k=0}^{\infty} \{k + .6\}. \end{cases}$$

Here, $\mu(t) < \frac{1}{2}$ for all $t \in \mathbb{T}$. From the previous example, we see that the allowable values are $\frac{1}{2} < a(t) < \frac{7}{2}$, which is satisfied for all $t \in \mathbb{T}$. For any t, the eigenvalues of the matrix $A^*(t)$ have a maximum value less than $-\frac{1}{2}$ when $\mu(t) < \frac{1}{2}$. As $\mu(t)$ decreases to 0, the maximum value decreases. Therefore, the maximum of all of the eigenvalues of the matrix $A^*(t)$ is always less than $-\frac{1}{2}$. So $A^*(t)$ is negative definite. Thus, we can set $\nu = \frac{1}{2}$. Checking that $-\frac{\nu}{\rho} = -\frac{1}{2} \in \mathcal{R}^+$, we now know that the norm of any solution $x(t)$ with initial value $x(t_0)$ is bounded above by the always positive decaying time scale exponential function $||x(t_0)||\sqrt{e_{-\frac{1}{2}}(t,t_0)}$. By letting $Q(t) = I$, the matrix $A^*(t)$ meets the criteria (i), (ii) in Theorem (4.2). Thus, the system above is uniformly exponentially stable.

4.4 Finding the Matrix $Q(t)$

First, we give a closed form for the unique, symmetric, and positive definite (when M is positive definite) solution matrix to the *time scale Lyapunov matrix equation*

$$A^T(t)Q(t) + Q(t)A(t) + \mu(t)A^T(t)Q(t)A(t) = -M, \qquad M = M^T. \tag{4.8}$$

We note that the time scale Lyapunov matrix equation is the unification (with $B(t) \equiv A^T(t)$) of the Sylvester matrix equation [5]

$$XA(t) + B(t)X = -M, \qquad M = M^T,$$

for the case $\mathbb{T} = \mathbb{R}$, and the Stein equation

$$B(t)XA(t) - X = -M, \qquad M = M^T,$$

for the case $\mathbb{T} = \mathbb{Z}$. The Stein matrix equation above is written assuming that one is using recursive form. It can easily be transformed into the equivalent difference form

$$XA(t) + B(t)X + B(t)XA(t) = -M, \qquad M = M^T.$$

To prove that the matrix $Q(t)$ is a solution to the time scale Lyapunov matrix equation (4.8), we first state the following theorem and corollary that can be found in [6].

Theorem 4.3. *Suppose $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and $C \in \mathbb{R}^{n\times n}$ is differentiable. If C is a solution of the matrix dynamic equation*

$$C^{\Delta} = A(\tau)C - C^{\sigma}A(\tau)$$

then

$$C(\tau)e_A(\tau, s) = e_A(\tau, s)C(s).$$

Corollary 4.1. *Suppose $A \in \mathcal{R}$ and C is a constant matrix. If C commutes with $A(t)$, then C commutes with $e_{A(t)}$. In particular, if $A(t)$ is constant matrix with respect to $e_{A(t)}$, then $A(t)$ commutes with $e_{A(t)}$.*

Now we present one of the main results of the dissertation.

Theorem 4.4 (Closed Form of the Matrix $Q(t)$). *If the $n \times n$ matrix $A(t)$ has all eigenvalues in the corresponding Hilger circle for every $t \geq t_0$, then for each $t \in \mathbb{T}$, there exists some time scale $\mathbb{S}$ such that integration over $I := [0, \infty)_{\mathbb{S}}$ yields a unique solution to (4.8) given by*

$$Q(t) = \int_I e_{A^T(t)}(s, 0)Me_{A(t)}(s, 0)\Delta s. \tag{4.9}$$

Moreover, if M is positive definite, then $Q(t)$ is positive definite for all $t \geq t_0$.

Proof. First, we fix an arbitrary $t \in \mathbb{T}$. Since all eigenvalues of $A(t)$ are in the corresponding Hilger circle, [40] shows (4.9) converges, so that $Q(t)$ is well defined. We now show for each fixed $t \in \mathbb{T}$, $Q(t)$ is a solution of (4.8).

Case I: $\mu(t) > 0$. Since $\mu(t)$ is a positive number, we define the time scale $\mathbb{S} = \mu(t)\mathbb{N}_0$. So for each $s \in \mathbb{S}$, we have that $\mu(s) \equiv \mu(t)$; in other words, for each fixed $t \in \mathbb{T}$, $\mathbb{S}$ has constant graininess. Substituting (4.9) and integrating over $I = [0, \infty)_{\mathbb{S}}$ we obtain

$$\begin{aligned}
A^T(t)Q(t) &+ Q(t)A(t) + \mu(t)A^T(t)Q(t)A(t) \\
&= \int_I A^T(t)e_{A^T(t)}(s,0)Me_{A(t)}(s,0)\Delta s \\
&\qquad + \int_I e_{A^T(t)}(s,0)Me_{A(t)}(s,0)A(t)\Delta s \\
&\qquad + \mu(t)\int_I A^T(t)e_{A^T(t)}(s,0)Me_{A(t)}(s,0)A(t)\Delta s \\
&= \int_I A^T(t)e_{A^T(t)}(s,0)Me_{A(t)}(s,0)[I + \mu(t)A(t)]\Delta s \\
&\qquad + \int_I e_{A^T(t)}(s,0)Me_{A(t)}(s,0)A(t)\Delta s \\
&= \int_I A^T(t)e_{A^T(t)}(s,0)M[I + \mu(t)A(t)]e_{A(t)}(s,0)\Delta s \\
&\qquad + \int_I e_{A^T(t)}(s,0)MA(t)e_{A(t)}(s,0)\Delta s.
\end{aligned}$$

But since $\mu(t) = \mu(s)$, this last line becomes

$$\begin{aligned}
&\int_I A^T(t)e_{A^T(t)}(s,0)M[I + \mu(s)A(t)]e_{A(t)}(s,0)\Delta s \\
&\qquad + \int_I e_{A^T(t)}(s,0)MA(t)e_{A(t)}(s,0)\Delta s \\
&= \int_I [e_{A^T(t)}(s,0)]^{\Delta_s} Me^{\sigma}_{A(t)}(s,0)\Delta s \\
&\qquad + \int_I e_{A^T(t)}(s,0)M[e_{A(t)}(s,0)]^{\Delta_s}\Delta s \\
&= \int_I [e_{A^T(t)}(s,0)Me_{A(t)}(s,0)]^{\Delta_s}\Delta s \\
&= e_{A^T(t)}(s,0)Me_{A(t)}(s,0)\Big|_0^{\infty} \\
&= -M.
\end{aligned}$$

Case II: $\mu(t) = 0$. Since $\mu(t) = 0$, we define the time scale $\mathbb{S} = \mathbb{R}$. Now substituting (4.9) and integrating over $I = [0, \infty)$ we obtain

$$\begin{aligned}
A^T(t)Q(t) &+ Q(t)A(t) + \mu(t)A^T(t)Q(t)A(t) \\
&= A^T(t)Q(t) + Q(t)A(t) \\
&= \int_I A^T(t)e_{A^T(t)}(s,0)Me_{A(t)}(s,0)\Delta s + \int_I e_{A^T(t)}(s,0)Me_{A(t)}(s,0)A(t)\Delta s \\
&= \int_I A^T(t)e^{A^T(t)\cdot s}Me^{A(t)\cdot s}ds + \int_I e^{A^T(t)\cdot s}Me^{A(t)\cdot s}A(t)ds \\
&= \int_I \frac{d}{ds}[e^{A^T(t)\cdot s}]Me^{A(t)\cdot s} + e^{A^T(t)\cdot s}M\frac{d}{ds}[e^{A(t)\cdot s}]ds \\
&= \int_I \frac{d}{ds}[e^{A^T(t)\cdot s}Me^{A(t)\cdot s}]ds \\
&= [e^{A^T(t)\cdot s}Me^{A(t)\cdot s}]\Big|_0^\infty \\
&= -M.
\end{aligned}$$

Since $t \in \mathbb{T}$ was arbitrary, but fixed, we see that $Q(t)$ defined as in (4.9) is a solution of (4.8) for each $t \in \mathbb{T}$.

Now, to show that $Q(t)$ is unique, suppose that $Q^*(t)$ is another solution to (4.8). Then

$$A^T(t)[Q^*(t) - Q(t)] + [Q^*(t) - Q(t)]A(t) + \mu(t)A^T(t)[Q^*(t) - Q(t)]A(t) = 0,$$

which implies

$$\begin{aligned}
e_{A^T(t)}(s,0)A^T(t)[Q^*(t) - Q(t)]e_{A(t)}(s,0) + e_{A^T(t)}(s,0)[Q^*(t) - Q(t)]A(t)e_{A(t)}(s,0) \\
+ \mu(t)e_{A^T(t)}(s,0)A^T(t)[Q^*(t) - Q(t)]A(t)e_{A(t)}(s,0) = 0, \qquad s \geq 0.
\end{aligned}$$

From this we obtain

$$[e_{A^T(t)}(s,0)[Q^*(t) - Q(t)]e_{A(t)}(s,0)]^{\Delta_s} = 0, \qquad s \geq 0. \tag{4.10}$$

Integrating both sides of (4.10) over $[0, \infty)_{\mathbb{S}}$, we have

$$[e_{A^T(t)}(s,0)[Q^*(t) - Q(t)]e_{A(t)}(s,0)]\Big|_0^\infty = -(Q^*(t) - Q(t)) = 0,$$

which implies that $Q^*(t) = Q(t)$.

Lastly, suppose that M is positive definite. Then $x^TMx > 0$, for all $n \times 1$ vectors $x \neq 0$. Clearly, $Q(t)$ is symmetric. To prove that $Q(t)$ is positive definite, we notice that for any nonzero $n \times 1$ vector x,

$$x^TQ(t)x = \int_I x^T e_{A^T(t)}(s,0) M e_{A(t)}(s,0) x \, \Delta s > 0,$$

which is true since M is positive definite. Hence, $Q(t)$ is positive definite. $\square$

Theorem 4.4 gives a closed form solution for the matrix equation (4.8). The next theorem offers a closed form solution for the matrix $Q(t)$ that satisfies the requirements of Theorem 4.2.

Theorem 4.5. *Let $\mathbb{T}$ be a time scale with bounded graininess, (i.e. $\mu_{\max} < \infty$). Suppose* (4.1) *is uniformly exponentially stable and there exists a positive constant α such that $||A(t)|| \leq \alpha$ for all t. Then*

$$Q(t) = \int_t^\infty \Phi_A^T(s,t)\Phi_A(s,t)\Delta s \tag{4.11}$$

satisfies the requirements of Theorem 4.2, where Φ_A is the transition matrix for the system (4.1).

Proof. First, to show that $Q(t)$ is well-defined, we need to show that the integral converges at each $t \in \mathbb{T}$. Since (4.1) is uniformly exponentially stable, we know that for some $\gamma, \lambda \in \mathbb{R}^+$ with $-\lambda \in \mathcal{R}^+$,

$$||\Phi_A(t,t_0)|| \leq \gamma e_{-\lambda}(t,t_0),$$

for every $t, t_0 \in \mathbb{T}$ with $t \geq t_0$.

This yields

$$\begin{aligned} ||\int_t^\infty \Phi_A^T(s,t)\Phi_A(s,t)\Delta s|| &\leq \int_t^\infty ||\Phi_A^T(s,t)|| \, ||\Phi_A(s,t)||\Delta s \\ &\leq \int_t^\infty (\gamma e_{-\lambda}(s,t))^2 \Delta s, \end{aligned}$$

which converges for all $t \in \mathbb{T}$. The value to which the last integral converges is also the value of ρ in Theorem 4.2.

Clearly $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}$ is symmetric at each t. We now show that there exist $\eta,\ \nu > 0$ so that the hypotheses in Theorem 4.2 are satisfied.

For ν, using the Leibniz rule for time scales [6], we differentiate (4.11) with respect to t, obtaining

$$
\begin{aligned}
Q^{\Delta}(t) &= \int_t^{\infty} \left[\Phi_A^T(s,t)\Phi_A(s,t)\right]^{\Delta_t} \Delta s - \Phi_A^T(t,\sigma(t))\Phi_A(t,\sigma(t)) \\
&= \int_t^{\infty} \Phi_A^T(s,t)^{\Delta_t}\Phi_A(s,\sigma(t)) + \Phi_A^T(s,t)\Phi_A(s,t)^{\Delta_t}\Delta s \\
&\qquad - \Phi_A^T(t,\sigma(t))\Phi_A(t,\sigma(t)) \\
&= \int_t^{\infty} \left[-\Phi_A(s,\sigma(t))A(t)\right]^T \Phi_A(s,\sigma(t)) - \Phi_A^T(s,t)\Phi_A(s,\sigma(t))A(t)\Delta s \\
&\qquad - \Phi_A^T(t,\sigma(t))\Phi_A(t,\sigma(t)) \\
&= \int_t^{\infty} -A^T(t)\Phi_A^T(s,\sigma(t))\Phi_A(s,\sigma(t)) - \Phi_A^T(s,t)\Phi_A(s,\sigma(t))A(t)\Delta s - \\
&\qquad \Phi_A^T(t,\sigma(t))\Phi_A(t,\sigma(t)) \\
&= \int_t^{\infty} -A^T(t)\Phi_{\ominus A^T}(\sigma(t),s)\Phi_{\ominus A^T}^T(\sigma(t),s) - \Phi_A^T(s,t)\Phi_{\ominus A^T}^T(\sigma(t),s)A(t)\Delta s \\
&\qquad - \Phi_{\ominus A^T}(\sigma(t),t)\Phi_{\ominus A^T}^T(\sigma(t),t) \\
&= -A^T(t)(I+\mu(t)A^T(t))^{-1}\int_t^{\infty} \Phi_{\ominus A^T}(t,s)\left[(I+\mu(t)A^T(t))^{-1}\Phi_{\ominus A^T}(t,s)\right]^T \Delta s \\
&\qquad - \int_t^{\infty} \Phi_A^T(s,t)\left[(I+\mu(t)A^T(t))^{-1}\Phi_{\ominus A^T}(t,s)\right]^T \Delta s A(t) \\
&\qquad - (I+\mu(t)A^T(t))^{-1}\Phi_{\ominus A^T}(t,t)\left[(I+\mu(t)A^T(t))^{-1}\Phi_{\ominus A^T}(t,t)\right]^T \\
&= -A^T(t)(I+\mu(t)A^T(t))^{-1}\int_t^{\infty} \Phi_{\ominus A^T}(t,s)\Phi_{\ominus A^T}^T(t,s)\Delta s(I+\mu(t)A(t))^{-1} \\
&\qquad - \int_t^{\infty} \Phi_A^T(s,t)\Phi_{\ominus A^T}^T(t,s)\Delta s(I+\mu(t)A(t))^{-1}A(t) \\
&\qquad - (I+\mu(t)A^T(t))^{-1}\Phi_{\ominus A^T}(t,t)\Phi_{\ominus A^T}^T(t,t)(I+\mu(t)A(t))^{-1} \\
&= -(I+\mu(t)A^T(t))^{-1}A^T(t)Q(t)(I+\mu(t)A(t))^{-1} \\
&\qquad - Q(t)A(t)(I+\mu(t)A^T(t))^{-1} - (I+\mu(t)A^T(t))^{-1}(I+\mu(t)A(t))^{-1}.
\end{aligned}
$$

Premultiplying both sides by $(I+\mu(t)A^T(t))$ and postmultiplying both sides by $(I+\mu(t)A(t))$, we obtain

$$(I+\mu(t)A^T(t))Q^\Delta(t)(I+\mu(t)A(t)) = -A^T(t)Q(t) - (I+\mu(t)A^T(t))Q(t)A(t) - I,$$

which is equivalent to

$$A^T(t)Q(t) + (I+\mu(t)A^T(t))Q(t)A(t) + (I+\mu(t)A^T(t))Q^\Delta(t)(I+\mu(t)A(t)) = -I.$$

So we set $\nu = 1$.

Lastly, we show that there exists an $\eta > 0$ such that $Q(t) \geq \eta I$, for all $t \in \mathbb{T}$. For any x,

$$\begin{aligned}
\left[x^T\Phi_A^T(s,t)\Phi_A(s,t)x\right]^{\Delta s} &= x^T\left[(A(s)\Phi_A(s,t))^T(I+\mu(s)A(s))\Phi_A(s,t)\right.\\
&\qquad \left.+\Phi_A^T(s,t)A(s)\Phi_A(s,t)\right]x\\
&= x^T\Phi_A^T(s,t)\left[A^T(s)+A(s)+\mu(s)A^T(s)A(s)\right]\Phi_A(s,t)x\\
&\geq -||A^T(s)+A(s)+\mu(s)A^T(s)A(s)||x^T\Phi_A^T(s,t)\Phi_A(s,t)x\\
&\geq -(2\alpha+\mu_{\max}\alpha^2)x^T\Phi_A^T(s,t)\Phi_A(s,t)x.
\end{aligned}$$

Since $\Phi_A \to 0$ as $s \to \infty$, we integrate both sides to obtain

$$\begin{aligned}
\int_t^\infty \left[x^T\Phi_A^T(s,t)\Phi_A(s,t)x\right]^{\Delta s}\Delta s &\geq -(2\alpha+\mu_{\max}\alpha^2)x^T\int_t^\infty \Phi_A^T(s,t)\Phi_A(s,t)\Delta s\, x\\
&= -(2\alpha+\mu_{\max}\alpha^2)x^TQ(t)x
\end{aligned}$$

which, after evaluating the integral, implies

$$-x^Tx \geq -(2\alpha+\mu_{\max}\alpha^2)x^TQ(t)x.$$

But this is of course equivalent to

$$Q(t) \geq \frac{1}{(2\alpha+\mu_{\max}\alpha^2)}I, \quad t \in \mathbb{T}.$$

So we set $\eta = \frac{1}{(2\alpha+\mu_{\max}\alpha^2)}$. $\square$

We remark that with the same hypotheses of Theorem 4.5, the more general form

$$Q(t) = \int_t^\infty \Phi_A^T(s,t) M \Phi_A(s,t) \Delta s$$

is a solution to matrix equation (4.3).

4.5 Slowly Varying Systems

The correct placement of eigenvalues in the complex plane of a time invariant system is necessary and sufficient to ensure the stability and/or exponential stability of the system. This is a well-known fact in the theory of differential equations and difference equations, and it is investigated in depth in the landmark paper on the stability of time invariant linear systems on time scales by Pötzsche, Siegmund, and Wirth [40].

However, eigenvalue placement alone is neither necessary nor sufficient for stability in the general time varying time scales case. Texts such as Brogan [7], Chen [9], and Rugh [42] give the following example of a time varying systems with “frozen” (time invariant) eigenvalues with negative real parts as well as bounded system matrices that still exhibit instability.

Example 4.3. Given the linear dynamic equation (4.1) with $t_0 = 0$ on the time scale $\mathbb{T} = \mathbb{R}$ and

$$A(t) = \begin{bmatrix} -1 + \alpha\cos^2(t) & 1 - \alpha\sin(t)\cos(t) \\ -1 - \alpha\sin(t)\cos(t) & -1 + \alpha\sin^2(t) \end{bmatrix},$$

where α is a positive constant, the pointwise eigenvalues are constants, given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}.$$

The transition matrix is given by

$$\Phi_A(t,0) = \begin{bmatrix} e^{(\alpha-1)t}\cos(t) & e^{-t}\sin(t) \\ e^{(\alpha-1)t}\sin(t) & e^{-t}\cos(t) \end{bmatrix}.$$

Thus, even though the pointwise eigenvalues have negative real parts with $0 < \alpha < 2$, the system exhibits unstable solutions when $\alpha > 1$.

The classic papers by Desoer [13], Rosenbrock [41], and a recent paper by Solo [46] demonstrate this fact for systems of differential equations as well, but they do show that under certain conditions, such as a bounded and sufficiently slowly varying system matrix, exponential stability can be obtained with correct eigenvalue placement in the complex plane. Desoer also published a similar paper [14] (a discrete analog to [13]) which illustrates the same instability characteristic of time varying systems in the discrete setting, but remedies the situation in essentially the same manner, with a bounded and sufficiently slowly varying system matrix.

To begin, we state a definition from Pötzsche, Siegmund, and Wirth's paper [40], in which the stability region for time invariant linear systems on time scales is introduced.

Definition 4.2. [40] The *regressive stability region* for the dynamic system (4.1) when $A(t) \equiv A$ is a constant is defined to be the set

$$\mathcal{S}(\mathbb{T}) = \left\{ \lambda \in \mathbb{C} : \limsup_{T \to \infty} \frac{1}{T - t_0} \int_{t_0}^{T} \lim_{s \searrow \mu(\tau)} \frac{\operatorname{Log}|1 + s\lambda|}{s} \Delta\tau < 0 \right\}.$$

It is easy to see that the regressive stability region is always contained in $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) < 0\}$. This definition essentially says if the time average of the constant $\lambda \in \mathbb{C}$ is negative and $1 + \mu(t)\lambda \neq 0$ for all $t \in \mathbb{T}^\kappa$, then λ resides in the set $\mathcal{S}(\mathbb{T})$. This definition is an important part of the requirement for exponential stability of a time invariant linear system on an arbitrary time scale. If $\lambda_i \in \mathcal{S}(\mathbb{T})$ for all $i = 1, \ldots, n$, along with a constant $\delta > 0$ such that $0 < \delta^{-1} \leq |1 + \mu(t)\lambda_i|$, for all $t \in \mathbb{T}^\kappa$, then the system (4.1), with $A(t) \equiv A$ constant, is uniformly exponentially stable, (i.e. there exists an $\alpha > 0$ such that for any $t_0 \in \mathbb{T}$, $\gamma > 0$ can be chosen independently of t_0 such that $||\Phi_A(t, t_0)|| \leq ||x(t_0)||\gamma e^{-\alpha(t - t_0)}$). The reader is referred to [40] for more explanation.

In the main theorem that follows, we require the eigenvalues $\lambda_i(t)$ of the time varying matrix $A(t)$ to satisfy $\mathrm{Re}_\mu[\lambda_i(t)] \leq -\varepsilon < 0$ for some $\varepsilon > 0$, all $t \in \mathbb{T}$, and all $i = 1, \ldots, n$, which is equivalent to all eigenvalues residing in the corresponding Hilger circle for all $t \in \mathbb{T}$ and $i = 1, \ldots, n$. Recall that the Hilger circle is defined as the set

$$\left\{\lambda \in \mathbb{C} : \left|\frac{1}{\mu(t)} + \lambda(t)\right| < \frac{1}{\mu(t)}\right\} \subset \mathcal{S}(\mathbb{T}).$$

Finally, we introduce the definition of the Kronecker product for use in Theorem 4.6. The Kronecker product allows the multiplication of any two matrices, regardless of the dimensions. This operation is an integral part of the theorem since it offers an unusual way to represent a matrix equation as a vector valued equation from which we can easily obtain bounds on the solution matrix.

Definition 4.3. The *Kronecker product* of the $n_A \times m_A$ matrix A and the $n_B \times m_B$ matrix B is the $n_A n_B \times m_A m_B$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m_A}B \\ \vdots & \ddots & \vdots \\ a_{n_A 1}B & \cdots & a_{n_A m_A}B \end{bmatrix}.$$

Some properties of the Kronecker product are contained in the following lemma [49].

Lemma 4.1. *Assume $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times n}$ with complex valued entries.*

(i) $(A \otimes I_n)(I_m \otimes B) = A \otimes B = (I_m \otimes B)(A \otimes I_n)$.

(ii) *If λ_i and γ_j are the eigenvalues for A and B respectively, with $i = 1, \ldots, m$ and $j = 1, \ldots, n$, then the eigenvalues of $A \otimes B$ are*

$$\lambda_i\gamma_j, \quad i = 1, \ldots, m,\ j = 1, \ldots, n,$$

and the eigenvalues of $(A \otimes I_n) + (I_m \otimes B)$ are

$$\lambda_i + \gamma_j, \quad i = 1, \ldots, m,\ j = 1, \ldots, n.$$

We now present the theorem for uniform exponential stability of slowly time varying systems which involves an eigenvalue condition on the time varying matrix $A(t)$ as well as the requirement that $A(t)$ is norm bounded and varies at a sufficiently slow rate (i.e. $||A^{\Delta}(t)|| \leq \beta$, for some positive constant β and all $t \in \mathbb{T}$).

Theorem 4.6 (Exponential Stability for Slowly Time Varying Systems). *Suppose for the regressive time varying linear dynamic system* (4.1) *with* $A(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *we have* $\mu_{\max}$, $\mu^{\Delta}_{\max} < \infty$, *there exists a constant* $\alpha > 0$ *such that* $||A(t)|| \leq \alpha$, *and there exists a constant* $0 < \varepsilon < \frac{1}{\mu_{\max}} \leq \frac{1}{\mu(t)}$ *such that for every pointwise eigenvalue* $\lambda_i(t)$ *of* $A(t)$, *the Hilger real part satisfies* $\mathrm{Re}_{\mu}[\lambda_i(t)] \leq -\varepsilon < 0$. *Then there exists a* $\beta > 0$ *such that if* $||A^{\Delta}(t)|| \leq \beta$, (4.1) *is uniformly exponentially stable.*

Proof. For each $t \in \mathbb{T}$, let $Q(t)$ be the solution of

$$A^T(t)Q(t) + Q(t)A(t) + \mu(t)A^T(t)Q(t)A(t) = -I. \tag{4.12}$$

By Theorem 4.4, existence, uniqueness, and positive definiteness of $Q(t)$ for each t is guaranteed. We also note that for each $t \in \mathbb{T}$, the solution of (4.12) is

$$Q(t) = \int_I e_{A^T(t)}(s, 0)e_{A(t)}(s, 0)\Delta s,$$

where $I := [0, \infty)_{\mathbb{S}}$ and $\mathbb{S} = \mu(t)\mathbb{N}_0$. For the remaining part of the proof, we show that $Q(t)$ can be used to satisfy the requirements of Theorem 4.2, so that uniform exponential stability of (4.1) follows. First, we use the Kronecker product and some of its properties to show the boundedness of the matrix $Q(t)$. We let e_i denote the i^{th} column of I, and $q_i(t)$ denote the i^{th} column of $Q(t)$. We then define the $n^2 \times 1$ vectors

$$e = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}, \qquad q(t) = \begin{bmatrix} q_1(t) \\ \vdots \\ q_n(t) \end{bmatrix}.$$

It can be computed to confirm that the $n \times n$ matrix equation (4.12) can be written as the $n^2 \times 1$ vector equation

$$\left[(A^T(t) \otimes I) + (I \otimes A^T(t)) + \mu(t)(A^T(t) \otimes A^T(t))\right] q(t) = -e. \tag{4.13}$$

We now prove that $q(t)$ is bounded above and that there exists a positive constant ρ such that $Q(t) \leq \rho I$, for all $t \in \mathbb{T}$. Since $A(t) \in \mathcal{R}$, this implies that the pointwise eigenvalues $\lambda_1(t), \ldots, \lambda_n(t)$ of $A(t)$ are also regressive. We also note that $I \in \mathcal{R}$. The pointwise eigenvalues of $A^T(t) \otimes I$ and $I \otimes A^T(t)$ are also $\lambda_1(t), \ldots, \lambda_n(t)$, by previously mentioned properties of the Kronecker product in Lemma 4.1. Because $(\mathcal{R}(\mathbb{T}, \mathbb{R}^{n^2 \times n^2}), \oplus)$ is a group we have that $(A^T(t) \otimes I)$, $(I \otimes A^T(t)) \in \mathcal{R}$ yields

$$\begin{aligned}
&(A^T(t) \otimes I) \oplus (I \otimes A^T(t)) \\
&\qquad = (A^T(t) \otimes I) + (I \otimes A^T(t)) + \mu(t)(A^T(t) \otimes I)(I \otimes A^T(t)) \\
&\qquad = (A^T(t) \otimes I) + (I \otimes A^T(t)) + \mu(t)(A^T(t) \otimes A^T(t)) \in \mathcal{R},
\end{aligned}$$

for all $t \in \mathbb{T}$.

Now, we show that $(A^T(t) \otimes I) \oplus (I \otimes A^T(t))$ has no eigenvalues equal to zero, so that $\det\left[(A^T(t) \otimes I) \oplus (I \otimes A^T(t))\right] \neq 0$. The n^2 pointwise eigenvalues of $(A^T(t) \otimes I) \oplus (I \otimes A^T(t)) = (A^T(t) \otimes I) + (I \otimes A^T(t)) + \mu(t)(A^T(t) \otimes A^T(t))$ are

$$\lambda_{i,j}(t) = \lambda_i(t) \oplus \lambda_j(t) = \lambda_i(t) + \lambda_j(t) + \mu(t)\lambda_i(t)\lambda_j(t) \in \mathcal{R},$$

for all $i, j = 1, \ldots, n$.

Recall that since $\mathrm{Re}_\mu[\lambda_i(t)] \leq -\varepsilon$ we have that $|1 + \mu(t)\lambda_i(t)| < 1$. Observe

$$\begin{aligned}
\mathrm{Re}_\mu[\lambda_i(t) \oplus \lambda_j(t)] &= \frac{|1 + \mu(t)(\lambda_i(t) \oplus \lambda_j(t))| - 1}{\mu(t)} \\
&= \frac{|(1 + \mu(t)\lambda_i(t))||(1 + \mu(t)\lambda_j(t))| - 1}{\mu(t)} \\
&< \frac{|(1 + \mu(t)\lambda_j(t))| - 1}{\mu(t)} \\
&= \mathrm{Re}_\mu[\lambda_j(t)] \\
&\leq -\varepsilon,
\end{aligned}$$

for all $t \in \mathbb{T}$ and all $i, j = 1, \ldots, n$. Therefore, $\mathrm{Re}_\mu[\lambda_i(t) \oplus \lambda_j(t)] < -\varepsilon < 0$ for $0 < \varepsilon < \frac{1}{\mu_{\max}} \leq \frac{1}{\mu(t)}$ and we also have the relationship $0 < \varepsilon \leq |\mathrm{Re}_\mu[\lambda_i(t) \oplus \lambda_j(t)]| \leq |\lambda_i(t) \oplus \lambda_j(t)|$.

Thus

$$\left|\det\left[(A^T(t) \otimes I) \oplus (I \otimes A^T(t))\right]\right| = \left|\prod_{i,j=1}^{n} [\lambda_i(t) \oplus \lambda_j(t)]\right| \geq \varepsilon^{n^2}, \quad t \in \mathbb{T}. \tag{4.14}$$

Now it is clear that $(A^T(t) \otimes I) \oplus (I \otimes A^T(t))$ is invertible at each $t \in \mathbb{T}$ since the determinant in (4.14) is nonzero and bounded away from zero for all t. Since $A(t)$ and $\mu(t)$ are bounded above, $A^T(t) \otimes I$ is bounded above, and hence the inverse

$$\left[(A^T(t) \otimes I) \oplus (I \otimes A^T(t))\right]^{-1}$$

is also bounded for all $t \in \mathbb{T}$. Since the right side of (4.13) is constant, we conclude that $q(t)$ is bounded for all $t \in \mathbb{T}$.

Clearly, $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ and $Q(t)$ is symmetric. Now we show that there exists a $\nu > 0$ such that

$$A^T(t)Q(t) + (I + \mu(t)A^T(t))Q(t)A(t) + (I + \mu(t)A(t))^T Q^\Delta(t)(I + \mu(t)A(t)) \leq -\nu I,$$

for all $t \in \mathbb{T}$. Since $Q(t)$ satisfies (4.12), the above inequality is equivalent to

$$(I + \mu(t)A(t))^T Q^\Delta(t)(I + \mu(t)A(t)) \leq (1 - \nu)I,$$

which gives

$$Q^\Delta(t) \leq (1 - \nu)(I + \mu(t)A^T(t))^{-1}(I + \mu(t)A(t))^{-1}. \tag{4.15}$$

Delta differentiating (4.12) with respect to t, we obtain

$$\begin{aligned} &A^{T^\sigma}(t)Q^\Delta(t) + A^{T^\Delta}(t)Q(t) + Q^\Delta(t)A^\sigma(t) + Q(t)A^\Delta(t) \\ &\quad + \mu^\Delta(t)A^T(t)Q(t)A(t) + \mu^\sigma(t)A^{T^\Delta}(t)Q(t)A(t) \\ &\quad + \mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)A(t) + \mu^\sigma(t)A^{T^\sigma}(t)Q^\sigma(t)A^\Delta(t) = 0. \end{aligned}$$

Recalling $Q^\sigma(t) = \mu(t)Q^\Delta(t) + Q(t)$, the equation above becomes

$$\begin{aligned}A^{T^\sigma}(t)Q^\Delta(t) &+ A^{T^\Delta}(t)Q(t) + Q^\Delta(t)A^\sigma(t) + Q(t)A^\Delta(t)\\ &+ \mu^\Delta(t)A^T(t)Q(t)A(t) + \mu^\sigma(t)A^{T^\Delta}(t)Q(t)A(t)\\ &+ \mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)A(t) + \mu(t)\mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)A^\Delta(t)\\ &+ \mu^\sigma(t)A^{T^\sigma}(t)Q(t)A^\Delta(t) = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&A^{T^\sigma}(t)Q^\Delta(t) + Q^\Delta(t)A^\sigma(t) + \mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)A(t) + \mu(t)\mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)A^\Delta(t)\\ &\quad= -A^{T^\Delta}(t)Q(t) - Q(t)A^\Delta(t) - \mu^\Delta(t)A^T(t)Q(t)A(t) - \mu^\sigma(t)A^{T^\Delta}(t)Q(t)A(t)\\ &\qquad - \mu^\sigma(t)A^{T^\sigma}(t)Q(t)A^\Delta(t).\end{aligned}$$

Transforming only the left hand side, we have

$$\begin{aligned}&A^{T^\sigma}(t)Q^\Delta(t) + Q^\Delta(t)A^\sigma(t) + \mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)A(t) + \mu(t)\mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)A^\Delta(t)\\ &\quad= A^{T^\sigma}(t)Q^\Delta(t) + Q^\Delta(t)A^\sigma(t) + \mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)(A(t) + \mu(t)A^\Delta(t))\\ &\quad= A^{T^\sigma}(t)Q^\Delta(t) + Q^\Delta(t)A^\sigma(t) + \mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)A^\sigma(t).\end{aligned}$$

Thus, we now have

$$\begin{aligned}&A^{T^\sigma}(t)Q^\Delta(t) + Q^\Delta(t)A^\sigma(t) + \mu^\sigma(t)A^{T^\sigma}(t)Q^\Delta(t)A^\sigma(t)\\ &\quad= -A^{T^\Delta}(t)Q(t) - Q(t)A^\Delta(t) - \mu^\Delta(t)A^T(t)Q(t)A(t) - \mu^\sigma(t)A^{T^\Delta}(t)Q(t)A(t)\\ &\qquad - \mu^\sigma(t)A^{T^\sigma}(t)Q(t)A^\Delta(t).\end{aligned} \tag{4.16}$$

For simplicity, let

$$\begin{aligned}X = A^{T^\Delta}(t)Q(t) &+ Q(t)A^\Delta(t) + \mu^\Delta(t)A^T(t)Q(t)A(t)\\ &+ \mu^\sigma(t)A^{T^\Delta}(t)Q(t)A(t) + \mu^\sigma(t)A^{T^\sigma}(t)Q(t)A^\Delta(t).\end{aligned}$$

Then the solution, $Q^\Delta(t)$, of the matrix equation (4.16) can be written as

$$Q^\Delta(t) = \int_{I^\sigma} e_{A^{T^\sigma}(t)}(s,0)Xe_{A^\sigma(t)}(s,0)\Delta s, \quad t \in \mathbb{T}^\kappa = \mathbb{T},$$

where $I^\sigma := [0,\infty)_{\mathbb{S}^\sigma}$ and $\mathbb{S}^\sigma = \mu^\sigma(t)\mathbb{N}_0$. To obtain a bound on $Q^\Delta(t)$, we use the boundedness of $Q(t), Q^\sigma(t)$, $A(t)$, $A^\Delta(t)$, $\mu_{\max}$, and $\mu^\Delta_{\max}$. For any $n \times 1$ vector x and any t,

$$
\begin{aligned}
&|x^T e_{A^{T\sigma}(t)}(s,0) X e_{A^\sigma(t)}(s,0)x| \\
&= |x^T e_{A^{T\sigma}(t)}(s,0)[A^{T^\Delta}(t)Q(t) + Q(t)A^\Delta(t) + \mu^\Delta(t)A^T(t)Q(t)A(t) \\
&\qquad + \mu^\sigma(t)A^{T^\Delta}(t)Q(t)A(t) + \mu^\sigma(t)A^{T^\sigma}(t)Q(t)A^\Delta(t)]e_{A^\sigma(t)}(s,0)x| \\
&\leq ||A^{T^\Delta}(t)Q(t) + Q(t)A^\Delta(t) + \mu^\Delta(t)A^T(t)Q(t)A(t) \\
&\qquad + \mu^\sigma(t)A^{T^\Delta}(t)Q(t)A(t) + \mu^\sigma(t)A^{T^\sigma}(t)Q(t)A^\Delta(t)||x^T e_{A^{T\sigma}(t)}(s,0)e_{A^\sigma(t)}(s,0)x.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|x^T Q^\Delta(t)x| &= \left| \int_{I^\sigma} x^T e_{A^{T\sigma}(t)}(s,0) X e_{A^\sigma(t)}(s,0)x\Delta s \right| \\
&\leq ||A^{T^\Delta}(t)Q(t) + Q(t)A^\Delta(t) + \mu^\Delta(t)A^T(t)Q(t)A(t) + \mu^\sigma(t)A^{T^\Delta}(t)Q(t)A(t) \\
&\qquad + \mu^\sigma(t)A^{T\sigma}(t)Q(t)A^\Delta(t)||x^T Q^\sigma(t)x \\
&\leq (2\beta||Q(t)|| + \mu^\Delta_{\max}\alpha^2||Q(t)|| + 2\mu_{\max}\alpha\beta||Q(t)||)x^T Q^\sigma(t)x \\
&= ||Q(t)||(2\beta + \alpha^2\mu^\Delta_{\max} + 2\alpha\beta\mu_{\max})x^T Q^\sigma(t)x.
\end{aligned}
$$

We now maximize the right hand side over all unit vectors x to obtain

$$|x^T Q^\Delta(t)x| \leq ||Q(t)||\;||Q^\sigma(t)||(2\beta + \alpha^2\mu^\Delta_{\max} + 2\alpha\beta\mu_{\max}),$$

and after maximizing the left hand side over all unit vectors x we conclude

$$||Q^\Delta(t)|| \leq ||Q(t)||\;||Q^\sigma(t)||(2\beta + \alpha^2\mu^\Delta_{\max} + 2\alpha\beta\mu_{\max}), \quad t \in \mathbb{T}^\kappa.$$

Using α, $\mu_{\max}$, $\mu^\Delta_{\max}$, and the norm bounds on $Q(t)$ and $Q^\sigma(t)$, the bound β on $||A^\Delta(t)||$ can be chosen so that we can create a bound for $Q^\Delta(t)$ which in turn yields a value for ν in (4.15).

Lastly, we show that there exists a positive constant η such that $\eta I \leq Q(t)$, for all $t \in \mathbb{T}$. For any t and any $n \times 1$ vector x,

$$
\begin{aligned}
&[x^T e_{A^T(t)}(s,0) e_{A(t)}(s,0) x]^{\Delta_s} \\
&\qquad = x^T [A^T(t) e_{A^T(t)}(s,0) e_{A(t)}(s,0) + e_{A^T(t)}(s,0) e_{A(t)}(s,0) A(t) \\
&\qquad\qquad + \mu(t) A^T(t) e_{A^T(t)}(s,0) e_{A(t)}(s,0) A(t)] x \\
&\qquad = x^T e_{A^T(t)}(s,0) [A^T(t) + A(t) + \mu(t) A^T(t) A(t)] e_{A(t)}(s,0) x \\
&\qquad \geq (-2\alpha - \mu_{\max}\alpha^2) x^T e_{A^T(t)}(s,0) e_{A(t)}(s,0) x.
\end{aligned}
$$

As $s \to \infty$, we know that $e_{A(t)}(s,0) \to 0$, so that

$$
-x^T x = \int_I [x^T e_{A^T(t)}(s,0) e_{A(t)}(s,0) x]^{\Delta_s} \Delta s \geq (-2\alpha - \mu_{\max}\alpha^2) x^T Q(t) x.
$$

But of course this is equivalent to

$$
Q(t) \geq \frac{1}{(2\alpha + \mu_{\max}\alpha^2)} I, \quad t \in \mathbb{T}.
$$

So we set $\eta = \frac{1}{(2\alpha + \mu_{\max}\alpha^2)}$. $\square$

4.6 Perturbation Results

It is also useful to consider state equations that are "close" (in an appropriate sense) to another linear state equation that is uniformly stable or uniformly exponentially stable. In Kalman and Bertram [30, 31], as well as Rugh [42], if the stability of the system (4.1) has already been determined by an appropriate Lyapunov function, then certain conditions on the perturbation matrix $F(t)$ guarantee stability of the perturbed linear system

$$
z^{\Delta}(t) = [A(t) + F(t)] z(t), \qquad z(t_0) = z_0. \tag{4.17}
$$

Motivated by these works, our aim is to prove analogous results for the general time scales case.

Theorem 4.7. *Suppose the linear state equation* (4.1) *is uniformly stable. Then there exists some* $\beta > 0$ *such that if*

$$\int_{\tau}^{\infty} ||F(s)||\Delta s \leq \beta$$

for all $\tau \in \mathbb{T}$, the perturbed linear dynamic equation (4.17) *is uniformly stable.*

Proof. For any t_0 and $z(t_0) = z_0$, by Theorem 2.7 the solution of (4.17) satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s)z(s)\Delta s,$$

where $\Phi_A(t, t_0)$ is the transition matrix for the system (4.1). By the uniform stability of (4.1), there exists a constant $\gamma > 0$ such that $||\Phi_A(t, \tau)|| \leq \gamma$, for all t, $\tau \in \mathbb{T}$ with $t \geq \tau$. Taking the norms of both sides, we see

$$||z(t)|| \leq \gamma||z_0|| + \int_{t_0}^{t} \gamma||F(s)||\, ||z(s)||\, \Delta s, \quad t \geq t_0.$$

Applying Gronwall's Inequality and a result in [16], we obtain

$$\begin{aligned} ||z(t)|| &\leq \gamma||z_0||e_{\gamma||F||}(t, t_0) \\ &= \gamma||z_0|| \exp\left(\int_{t_0}^{t} \frac{\operatorname{Log}(1 + \mu(s)\gamma||F(s)||)}{\mu(s)}\Delta s\right) \\ &\leq \gamma||z_0|| \exp\left(\int_{t_0}^{\infty} \frac{\operatorname{Log}(1 + \mu(s)\gamma||F(s)||)}{\mu(s)}\Delta s\right) \\ &\leq \gamma||z_0|| \exp\left(\int_{t_0}^{\infty} \gamma||F(s)||\Delta s\right) \\ &\leq \gamma||z_0||e^{\gamma\beta}, \quad t \geq t_0. \end{aligned}$$

Since γ can be used for any t_0 and $z(t_0) = z_0$, the state equation (4.17) is uniformly stable. □

Theorem 4.8. *Suppose the linear state equation* (4.1) *is uniformly exponentially stable* (*i.e.* $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ *for some constants* λ, $\gamma > 0$ *with* $-\lambda \in \mathcal{R}^+$) *and the exponential decay factor $-\lambda$ is uniformly regressive on the time scale $\mathbb{T}$. Then there exists some $\beta > 0$ such that if*

$$||F(t)|| \leq \beta \tag{4.18}$$

for all $t \geq t_0$ with $t, t_0 \in \mathbb{T}$, the perturbed linear dynamic equation (4.17) *is uniformly exponentially stable.*

Proof. For any t_0 and $z(t_0) = z_0$, by Theorem 2.7 the solution of (4.17) satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s)z(s)\Delta s,$$

where $\Phi_A(t, t_0)$ is the transition matrix for the system (4.1). By the uniform exponential stability of (4.1), there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||\Phi_A(t, \tau)|| \leq \gamma e_{-\lambda}(t, \tau)$, for all t, $\tau \in \mathbb{T}$ with $t \geq \tau$. By taking the norms of both sides, we have

$$||z(t)|| \leq \gamma e_{-\lambda}(t, t_0)||z_0|| + \int_{t_0}^{t} \gamma e_{-\lambda}(t, \sigma(s))||F(s)||\, ||z(s)||\, \Delta s, \quad t \geq t_0.$$

Rearranging and applying the uniform regressivity bound $0 < \delta^{-1} \leq (1 - \mu(t)\lambda)$, for some $\delta > 0$ and all $t \in \mathbb{T}$, and the inequality (4.18),

$$\begin{aligned} e_{-\lambda}(t_0, t)||z(t)|| &\leq \gamma||z_0|| + \int_{t_0}^{t} \gamma||F(s)||e_{-\lambda}(t_0, s)e_{-\lambda}(s, \sigma(s))||z(s)||\, \Delta s \\ &\leq \gamma||z_0|| + \int_{t_0}^{t} \gamma\beta(1 - \mu(s)\lambda)^{-1}e_{-\lambda}(t_0, s)||z(s)||\, \Delta s \\ &\leq \gamma||z_0|| + \int_{t_0}^{t} \gamma\beta\delta e_{-\lambda}(t_0, s)||z(s)||\, \Delta s, \quad t \geq t_0. \end{aligned}$$

Defining $\psi(t) := e_{-\lambda}(t_0, t)||z(t)||$, we now have

$$\psi(t) \leq \gamma||z_0|| + \int_{t_0}^{t} \gamma\beta\delta\psi(s)\, \Delta s, \quad t \geq t_0.$$

By Gronwall's Inequality, we obtain

$$\psi(t) \leq \gamma||z_0||e_{\gamma\beta\delta}(t, t_0), \quad t \geq t_0.$$

Thus, substituting back in for $\psi(t)$, we conclude

$$||z(t)|| \leq \gamma||z_0||e_{\gamma\beta\delta}(t, t_0)e_{-\lambda}(t, t_0) = e_{-\lambda\oplus\gamma\beta\delta}(t, t_0), \quad t \geq t_0.$$

We need $-\lambda \oplus \gamma\beta\delta \in \mathcal{R}^+$ and negative for all $t \in \mathbb{T}$. Observe, since $\gamma\beta\delta > 0$, it is positively regressive, and so $\gamma\beta\delta \in \mathcal{R}^+$. Since $\mathcal{R}^+$ is a subgroup of $\mathcal{R}$, we see that

$-\lambda \oplus \gamma\beta\delta \in \mathcal{R}^+$. So we must have

$$-\lambda < -\lambda \oplus \gamma\beta\delta < 0$$
$$-\lambda < -\lambda + \gamma\beta\delta - \mu(t)\lambda\gamma\beta\delta < 0$$
$$0 < \gamma\beta\delta - \mu(t)\lambda\gamma\beta\delta < \lambda$$
$$0 < \gamma\beta\delta(1 - \mu(t)\lambda) < \lambda$$
$$0 < \beta < \frac{\lambda}{\gamma\delta(1 - \mu(t)\lambda)}$$

for all $t \in \mathbb{T}$. Thus, by choosing β accordingly and since γ is independent of t_0 and $z(t_0) = z_0$, the state equation (4.17) is uniformly exponentially stable. □

In the following theorem, we show that under certain conditions on the linear and nonlinear perturbations, the resulting perturbed nonlinear initial value problem will still yield uniformly exponentially stable solutions.

Theorem 4.9. *Given the nonlinear regressive initial value problem*

$$x^{\Delta}(t) = [A(t) + F(t)]\, x(t) + g(t, x(t)), \qquad x(t_0) = x_0, \tag{4.19}$$

and an arbitrary time scale $\mathbb{T}$*, suppose* (4.1) *is uniformly exponentially stable* (*i.e.* $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ *for some constants* λ, $\gamma > 0$ *with* $-\lambda \in \mathcal{R}^+$) *and the exponential decay factor* $-\lambda$ *is uniformly regressive on the time scale* $\mathbb{T}$*, the matrix* $F(t) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *satisfies* $||F(t)|| \leq \beta$ *for all* $t \in \mathbb{T}$*, the vector-valued function* $g(t, x(t)) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$ *satisfies* $||g(t, x(t))|| \leq \epsilon||x(t)||$ *for all* $t \in \mathbb{T}$ *and* $x(t)$*, and the solution* $x(t) \in \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R}^n)$ *is defined for all* $t \geq t_0$*. Then if* β *and* ϵ *are sufficiently small, there exist constants* γ, $\lambda^* > 0$ *with* $-\lambda^* \in \mathcal{R}^+$ *such that*

$$||x(t)|| \leq \gamma||x_0||e_{-\lambda^*}(t, t_0)$$

for all $t \geq t_0$.

Proof. Observe that the solution to (4.19) is given by

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))[F(s)x(s) + g(s, x(s))]\Delta s, \tag{4.20}$$

for all $t \geq t_0$. Since (4.1) is uniformly exponentially stable, there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ for all $t \geq t_0$. Recall $||F(t)|| \leq \beta$, $||g(t, x(t))|| \leq \epsilon ||x(t)||$ for all $t \in \mathbb{T}$, and since the decay factor $-\lambda$ is uniformly regressive on $\mathbb{T}$, there exists a $\delta > 0$ such that $0 < \delta^{-1} \leq (1 - \mu(t)\lambda)$ for all $t \in \mathbb{T}$ which implies that $0 < (1 - \mu(t)\lambda)^{-1} \leq \delta$. Taking the norms of both sides of (4.20), we obtain

$$\begin{aligned}
||x(t)|| &\leq ||\Phi_A(t, t_0)||\ ||x_0|| + \int_{t_0}^{t} ||\Phi_A(t, \sigma(s))||(||F(s)||\ ||x(s)|| + ||g(s, x(s))||)\Delta s \\
&\leq \gamma e_{-\lambda}(t, t_0)\ ||x_0|| + \int_{t_0}^{t} \gamma e_{-\lambda}(t, \sigma(s))(\beta||x(s)|| + \epsilon||x(s)||)\Delta s \\
&\leq e_{-\lambda}(t, t_0)\left[\gamma||x_0|| + \int_{t_0}^{t} \gamma e_{-\lambda}(t_0, \sigma(s))(\beta + \epsilon)||x(s)||\Delta s\right] \\
&= e_{-\lambda}(t, t_0)\left[\gamma||x_0|| + \int_{t_0}^{t} \gamma(\beta + \epsilon)e_{-\lambda}(t_0, s)e_{-\lambda}(s, \sigma(s))||x(s)||\Delta s\right] \\
&= e_{-\lambda}(t, t_0)\left[\gamma||x_0|| + \int_{t_0}^{t} \gamma(\beta + \epsilon)e_{-\lambda}(t_0, s)(1 - \mu(s)\lambda)^{-1}||x(s)||\Delta s\right] \\
&\leq e_{-\lambda}(t, t_0)\left[\gamma||x_0|| + \int_{t_0}^{t} \gamma(\beta + \epsilon)e_{-\lambda}(t_0, s)\delta||x(s)||\Delta s\right] \\
&= e_{-\lambda}(t, t_0)\left[\gamma||x_0|| + \int_{t_0}^{t} \gamma\delta(\beta + \epsilon)e_{-\lambda}(t_0, s)||x(s)||\Delta s\right],
\end{aligned}$$

for all $t \geq t_0$. Define $\psi(t) := e_{-\lambda}(t_0, t)||x(t)||$. We now have

$$\psi(t) \leq \gamma||x_0|| + \int_{t_0}^{t} \gamma\delta(\beta + \epsilon)\psi(s)\Delta s,$$

and by Gronwall's inequality,

$$\psi(t) \leq \gamma||x_0||e_{\gamma\delta(\beta+\epsilon)}(t, t_0).$$

Substituting back in for $\psi(t)$ this implies

$$||x(t)|| \leq \gamma||x_0||e_{\gamma\delta(\beta+\epsilon)}(t, t_0)e_{-\lambda}(t, t_0) = \gamma||x_0||e_{-\lambda\oplus\gamma\delta(\beta+\epsilon)}(t, t_0).$$

To conclude, we need $-\lambda \oplus \gamma\delta(\beta+\epsilon) \in \mathcal{R}^+$ and at the same time $-\lambda \oplus \gamma\delta(\beta+\epsilon) < 0$. Observe that $\gamma\delta(\beta+\epsilon) > 0$ implies $\gamma\delta(\beta+\epsilon) \in \mathcal{R}^+$ and since $\mathcal{R}^+$ is a subgroup of $\mathcal{R}$, we have $-\lambda \oplus \gamma\delta(\beta+\epsilon) \in \mathcal{R}^+$. So we need

$$\begin{aligned}
-\lambda &< -\lambda \oplus \gamma\delta(\beta+\epsilon) < 0 \\
-\lambda &< -\lambda + \gamma\delta(\beta+\epsilon) - \mu(t)\lambda\gamma\delta(\beta+\epsilon) < 0 \\
0 &< \gamma\delta(\beta+\epsilon) - \mu(t)\lambda\gamma\delta(\beta+\epsilon) < \lambda \\
0 &< (1-\mu(t)\lambda)\gamma\delta(\beta+\epsilon) < \lambda \\
0 &< \beta < \frac{\lambda}{(1-\mu(t)\lambda)\gamma\delta} - \epsilon.
\end{aligned}$$

From this result, we must have $\frac{\lambda}{(1-\mu(t)\lambda)\gamma\delta} - \epsilon > 0$ for all $t \in \mathbb{T}$, i.e. $\epsilon < \frac{\lambda}{(1-\mu(t)\lambda)\gamma\delta}$ for all $t \in \mathbb{T}$.

Thus, to fulfill the requirements of the theorem, we must satisfy the following:

$$0 < \epsilon < \frac{\lambda}{(1-\mu(t)\lambda)\gamma\delta}, \quad 0 < \beta < \frac{\lambda}{(1-\mu(t)\lambda)\gamma\delta} - \epsilon, \quad \text{and} \quad -\lambda^* := -\lambda \oplus \gamma\delta(\beta+\epsilon)$$

for all $t \in \mathbb{T}$. □

Corollary 4.2. *Given the nonlinear regressive initial value problem* (4.19) *with* $A(t) \equiv A$ *a constant matrix, suppose* $\mathrm{spec}(A) \in \mathcal{S}(\mathbb{T})$ *for all* $t \in \mathbb{T}$*, the exponential decay factor in uniformly regressive, the matrix* $F(t) \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *satisfies* $||F(t)|| \leq \beta$ *for all* $t \in \mathbb{T}$*, the vector-valued function* $g(t, x(t)) \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ *satisfies* $||g(t, x(t))|| \leq \epsilon||x(t)||$ *for all* $t \in \mathbb{T}$ *and* $x(t)$*, and the solution* $x(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ *is defined for all* $t \geq t_0$*. Then if* β *and* ϵ *are sufficiently small, there exist constants* γ*,* $\lambda^* > 0$ *with* $-\lambda^* \in \mathcal{R}^+$ *such that*

$$||x(t)|| \leq \gamma||x_0||e_{-\lambda^*}(t, t_0)$$

for all $t \geq t_0$*.*

Proof. The proof follows exactly as in Theorem 4.9, with the observation that $\Phi_A(t, t_0) \equiv e_A(t, t_0)$, so the solution to (4.1) with $A(t) \equiv A$ is $x(t) = e_A(t, t_0)x_0$ and thus we now have the bound $||\Phi_A(t, t_0)|| = ||e_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$, for some constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$. □

4.7 Instability Criterion

We can also employ the unified time scale quadratic Lyapunov function to determine when the system (4.1) is unstable. This is a very useful result when the development of a suitable matrix $Q(t)$ is difficult and the possibility of an unstable system arises. In the next theorem, we develop one type of instability criterion.

Theorem 4.10. *Suppose there exists an $n \times n$ matrix $Q(t) \in \mathrm{C}^1_{\mathrm{rd}}$ that is symmetric for all $t \in \mathbb{T}$ and has the following two properties*

(i) $||Q(t)|| \leq \rho$,

(ii) $A^T(t)Q(t) + (I + \mu(t)A^T(t))(Q^\Delta(t) + Q(t)A(t) + \mu(t)Q^\Delta(t)A(t)) \leq -\nu I$,

where $\rho, \nu > 0$. Also suppose that there exists some $t^* \in \mathbb{T}$ such that $Q(t^*)$ is not positive semidefinite. Then the linear dynamic equation (4.1) *is not uniformly stable.*

Proof. Suppose that $x(t)$ is the solution of (4.1) with initial conditions $t_0 = t^*$ and $x(t_0) = x(t^*) = x_0$ with $x_0^T Q(t^*) x_0 < 0$. Then

$$\begin{aligned} x^T(t)Q(t)x(t) - x_0^T Q(t_0)x_0 &= \int_{t_0}^{t} \left[x^T(s)Q(s)x(s)\right]^{\Delta s} \Delta s \\ &\leq -\nu \int_{t_0}^{t} x^T(s)x(s)\Delta s \leq 0, \quad t \geq t_0. \end{aligned}$$

From this inequality, we see

$$x^T(t)Q(t)x(t) \leq x_0^T Q(t_0)x_0 < 0, \quad t \geq t_0.$$

By condition (ii),

$$-\rho||x(t)||^2 \leq x^T(t)Q(t)x(t) \leq x^T(t_0)Q(t_0)x(t_0) < 0, \quad t \geq t_0,$$

which leads to

$$||x(t)||^2 \geq \frac{1}{\rho}|x^T(t)Q(t)x(t)| > 0, \quad t \geq t_0. \tag{4.21}$$

Again by condition (ii),

$$\nu \int_{t_0}^{t} x^T(s)x(s)\Delta s \leq x_0^T Q(t_0)x_0 - x^T(t)Q(t)x(t)$$

$$\leq |x_0^T Q(t_0)x_0| + |x^T(t)Q(t)x(t)|$$

$$\leq 2|x^T(t)Q(t)x(t)|, \quad t \geq t_0.$$

Using (4.10), we obtain

$$\int_{t_0}^{t} x^T(s)x(s)\Delta s \leq \frac{2\rho}{\nu}||x(t)||^2, \quad t \geq t_0. \tag{4.22}$$

Finally, we show that $x(t)$ is unbounded and hence (4.1) is not uniformly stable. To this end, suppose there exists some $\gamma > 0$ so that $||x(t)|| \leq \gamma$ for all $t \geq t_0$. Then (4.22) implies

$$\int_{t_0}^{t} x^T(s)x(s)\Delta s \leq \frac{2\rho\gamma^2}{\nu}, \quad t \geq t_0.$$

By this last inequality, $||x(t)|| \to 0$ as $t \to \infty$, which contradicts (4.21). Thus, the solution $x(t)$ cannot be bounded, which shows that (4.1) is not uniformly stable. $\square$

CHAPTER FIVE

The Lyapunov Transformation and Stability

We begin by analyzing the stability preserving property associated with a change of variables using a Lyapunov transformation on the regressive time varying linear dynamic system

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0. \tag{5.1}$$

Definition 5.1. A *Lyapunov transformation* is an invertible matrix $L(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ with the property that, for some positive $\eta, \rho \in \mathbb{R}$,

$$||L(t)|| \leq \rho \qquad \text{and} \qquad \det L(t) \geq \eta \tag{5.2}$$

for all $t \in \mathbb{T}$.

The two following lemmas can be found in the classic text by Aitken [2].

Lemma 5.1. *Suppose that $A(t)$ is an $n \times n$ matrix such that $A^{-1}(t)$ exists for all $t \in \mathbb{T}$. If there exists a constant $\alpha > 0$ such that $||A^{-1}(t)|| \leq \alpha$ for each t, then there exists a constant β such that $|\det A(t)| \geq \beta$ for all $t \in \mathbb{T}$.*

Lemma 5.2. *Suppose that $A(t)$ is an $n \times n$ matrix such that $A^{-1}(t)$ exists for all $t \in \mathbb{T}$. Then*

$$||A^{-1}(t)|| \leq \frac{||A(t)||^{n-1}}{|\det A(t)|}$$

for all $t \in \mathbb{T}$.

A consequence of Lemma 5.1 and Lemma 5.2 is that the inverse of a Lyapunov transformation is also bounded. An equivalent condition to (5.2) is that there exists a $\rho > 0$ such that

$$||L(t)|| \leq \rho \qquad \text{and} \qquad ||L^{-1}(t)|| \leq \rho \tag{5.3}$$

for all $t \in \mathbb{T}$.

Theorem 5.1. *Suppose that* $L(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$*, with* $L(t)$ *invertible for all* $t \in \mathbb{T}$ *and* $A(t)$ *is from the linear dynamic system* (5.1)*. Then the transition matrix for the system*

$$Z^{\Delta}(t) = G(t)Z(t), \quad Z(\tau) = I \tag{5.4}$$

where

$$G(t) = L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t) \tag{5.5}$$

is given by

$$\Phi_G(t, \tau) = L^{-1}(t)\Phi_A(t, \tau)L(\tau), \tag{5.6}$$

for any $t, \tau \in \mathbb{T}$*.*

Proof. First we see that by definition, $G(t) \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$. For any $\tau \in \mathbb{T}$, we define

$$X(t) = L^{-1}(t)\Phi_A(t, \tau)L(\tau). \tag{5.7}$$

It is obvious that for $t = \tau$, $X(\tau) = I$. Temporarily rearranging (5.7) by multiplying by $L(t)$ on both sides and differentiating $L(t)X(t)$ with respect to t, we obtain [6, Thm. 5.3 (iv)]

$$L^{\Delta}(t)X(t) + L^{\sigma}(t)X^{\Delta}(t) = \Phi_A^{\Delta}(t, \tau)L(\tau) = A(t)\Phi_A(t, \tau)L(\tau),$$

and thus

$$\begin{aligned}
L^{\sigma}(t)X^{\Delta}(t) &= A(t)\Phi_A(t, \tau)L(\tau) - L^{\Delta}(t)X(t) \\
&= A(t)\Phi_A(t, \tau)L(\tau) - L^{\Delta}(t)L^{-1}(t)\Phi_A(t, \tau)L(\tau) \\
&= [A(t) - L^{\Delta}(t)L^{-1}(t)]\Phi_A(t, \tau)L(\tau).
\end{aligned}$$

Multiplying both sides by $L^{\sigma^{-1}}(t)$ and noting (5.5) and (5.7),

$$\begin{aligned}
X^{\Delta}(t) &= [L^{\sigma^{-1}}(t)A(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t)L^{-1}(t)]\Phi_A(t, \tau)L(\tau) \\
&= [L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t)]L^{-1}(t)\Phi_A(t, \tau)L(\tau) \\
&= G(t)X(t).
\end{aligned}$$

Since this is valid for any $\tau \in \mathbb{T}$, (5.6) is the transition matrix for (5.4). Additionally, if the initial value specified in (5.4) was not the identity matrix, i.e. $Z(t_0) = Z_0 \neq I$, then the solution is $X(t) = \Phi_G(t, \tau)Z_0$. $\square$

5.1 *Preservation of Uniform Stability*

Theorem 5.2. *Suppose that* $z(t) = L^{-1}(t)x(t)$ *is a Lyapunov transformation. Then the system* (5.1) *is uniformly stable if and only if*

$$z^{\Delta}(t) = \left[L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t)\right] z(t), \quad z(t_0) = z_0, \tag{5.8}$$

is uniformly stable.

Proof. Equations (5.1) and (5.8) are related by the change of variables $z(t) = L^{-1}(t)x(t)$. By Theorem 5.1, the relationship between the two transition matrices is

$$\Phi_G(t, t_0) = L^{-1}(t)\Phi_A(t, t_0)L(t_0).$$

Suppose that (5.1) is uniformly stable. Then there exists a $\gamma > 0$ such that $||\Phi_A(t, t_0)|| \leq \gamma$ for all $t, t_0 \in \mathbb{T}$ with $t \geq t_0$. By Lemma 5.2, with η and ρ as in (5.2) and (5.3), we have

$$\begin{aligned} ||\Phi_G(t, t_0)|| &= ||L^{-1}(t)\Phi_A(t, t_0)L(t_0)|| \\ &\leq ||L^{-1}(t)||\ ||\Phi_A(t, t_0)||\ ||L(t_0)|| \\ &\leq \frac{\gamma\rho^n}{\eta} = \gamma_G, \end{aligned}$$

for all $t, t_0 \in \mathbb{T}$ with $t \geq t_0$. By Theorem 3.1, since $||\Phi_G(t, t_0)|| \leq \gamma_G$, the system (5.8) is uniformly stable. The converse is similar. $\square$

5.2 *Preservation of Uniform Exponential Stability*

Theorem 5.3. *Suppose that* $z(t) = L^{-1}(t)x(t)$ *is a Lyapunov transformation. Then the system* (5.1) *is uniformly exponentially stable if and only if*

$$z^{\Delta}(t) = \left[L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t) \right] z(t), \quad z(t_0) = z_0, \tag{5.9}$$

is uniformly exponentially stable.

Proof. Equations (5.1) and (5.9) are related by the change of variables $z(t) = L^{-1}(t)x(t)$. By Theorem 5.1, the relationship between the two transition matrices is

$$\Phi_G(t, t_0) = L^{-1}(t)\Phi_A(t, t_0)L(t_0).$$

Suppose that (5.1) is uniformly exponentially stable. Then there exist constants $\lambda, \gamma > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ for all $t \geq t_0$ with $t, t_0 \in \mathbb{T}$. Then by Lemma 5.2, with η and ρ as in (5.2) and (5.3), we have

$$\begin{aligned} ||\Phi_G(t, t_0)|| &= ||L^{-1}(t)\Phi_A(t, t_0)L(t_0)|| \\ &\leq ||L^{-1}(t)|| \, ||\Phi_A(t, t_0)|| \, ||L(t_0)|| \\ &\leq \frac{\gamma\rho^n}{\eta} e_{-\lambda}(t, t_0) = \gamma_G e_{-\lambda}(t, t_0), \end{aligned}$$

for all $t, t_0 \in \mathbb{T}$ with $t \geq t_0$.

By Theorem 3.2, since $||\Phi_G(t, t_0)|| \leq \gamma_G e_{-\lambda}(t, t_0)$, the system (5.9) is uniformly exponentially stable. The converse is similar. □

Corollary 5.1. *Suppose that* $z(t) = L^{-1}(t)x(t)$ *is a Lyapunov transformation. Then the system* (5.1) *is uniformly asymptotically stable if and only if*

$$z^{\Delta}(t) = \left[L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t) \right] z(t), \quad z(t_0) = z_0 \tag{5.10}$$

is uniformly asymptotically stable.

Proof. Suppose (5.10) is uniformly asymptotically stable. By Theorem 3.4, (5.10) is uniformly exponentially stable and Theorem 5.3 now implies that (5.1) is uniformly exponentially stable. Thus, by Theorem 3.4, the linear dynamic system (5.1) is uniformly exponentially stable if and only if it is uniformly asymptotically stable. The converse is similar. □

CHAPTER SIX

Floquet Theory

We begin with definitions that will be used throughout the remainder of the dissertation.

Definition 6.1. Let $p \in [0, \infty)$. Then the time scale $\mathbb{T}$ is *p-periodic* if we have the following:

(i) $t \in \mathbb{T}$ implies that $t + p \in \mathbb{T}$,

(ii) $\mu(t) = \mu(t + p)$,

for all $t \in \mathbb{T}$.

Definition 6.2. An $n \times n$-matrix valued function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is *p-periodic* if $A(t) = A(t + p)$ for all $t \in \mathbb{T}$.

We will henceforth assume that the time scale we are working with is p-periodic.

6.1 The Homogeneous Equation

We consider the regressive time varying linear dynamic initial value problem

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0 \tag{6.1}$$

where $A(t) \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and p-periodic for all $t \in \mathbb{T}$.

We note that in general, it is not necessary that the period of $A(t)$ be equal to the period of the time scale on which the system is being analyzed. We let the period of the time scale and the function $A(t)$ be equal for simplicity.

Lemma 6.1. *Suppose that $\mathbb{T}$ is a p-periodic time scale and $R \in \mathcal{R}(\mathbb{T}, \mathbb{C}^{n\times n})$. Then the solution of the regressive dynamic matrix initial value problem*

$$Z^{\Delta}(t) = RZ(t), \qquad Z(t_0) = Z_0, \tag{6.2}$$

is unique up to a period p shift. That is, $e_R(t, t_0) = e_R(t + kp, t_0 + kp)$*, for all* $t \in \mathbb{T}$ *and* $k \in \mathbb{N}_0$*.*

Proof. By [6], the unique solution to (6.2) is $e_R(t, t_0)Z_0$. Observe

$$e_R^{\Delta}(t, t_0)Z_0 = R\, e_R(t, t_0)Z_0,$$

and

$$e_R(t, t_0)|_{t=t_0} Z_0 = e_R(t_0, t_0)Z_0 = Z_0.$$

Now we show that $e_R(t, t_0) = e_R(t + kp, t_0 + kp)$. We show this by observing that $e_R(t + kp, t_0 + kp)Z_0$ also solves the matrix initial value problem (6.2). We see

$$e_R^{\Delta_{t+kp}}(t + kp, t_0 + kp)Z_0 = R\, e_R(t + kp, t_0 + kp)$$

and

$$e_R(t + kp, t_0 + kp)|_{t+kp=t_0+kp} = e_R(t + kp, t_0 + kp)|_{t=t_0} = e_R(t_0 + kp, t_0 + kp)Z_0 = Z_0.$$

By [6], we have uniqueness of solutions to the matrix IVP (6.2). Thus,

$$e_R(t + kp, t_0 + kp) = e_R(t, t_0), \quad \text{for all } t \in \mathbb{T} \text{ and } k \in \mathbb{N}_0.$$

Therefore, e_R can be shifted by integer multiples of p. □

The next theorem is the unified and extended time scale version of the Floquet decomposition for p-periodic time varying linear dynamic systems.

Theorem 6.1 (The Unified Floquet Decomposition for Time Scales). *Suppose that there exists an* $n \times n$ *constant regressive matrix* R *such that* $e_R(p + t_0, t_0) = \Phi_A(p + t_0, t_0)$*, where* Φ_A *is the transition matrix for the p-periodic system* (6.1)*. Then the transition matrix for* (6.1) *can be written in the form*

$$\Phi_A(t, \tau) = L(t)e_R(t, \tau)L^{-1}(\tau) \quad \textit{for all } t, \tau \in \mathbb{T}, \tag{6.3}$$

where $R \in \mathbb{C}^{n\times n}$ *is a constant matrix and* $L(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *is a p-periodic Lyapunov transformation. We refer to* (6.3) *as the Floquet decomposition for* Φ_A*.*

Proof. We begin by defining the constant matrix R as the solution to the equation

$$e_R(p+t_0,t_0)=\Phi_A(p+t_0,t_0),$$

which may require either taking the natural logarithm or obtaining the invertible p^{th} root of the real-valued invertible constant matrix $\Phi_A(p+t_0,t_0)$. Thus, it is possible that a complex R is obtained. Define the matrix $L(t)$ by

$$L(t)=\Phi_A(t,t_0)e_R^{-1}(t,t_0). \tag{6.4}$$

It follows from this definition that $L(t)\in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^{n\times n})$ and is invertible at each $t\in\mathbb{T}$.

It is easily seen that

$$\Phi_A(t,t_0)=L(t)e_R(t,t_0)$$

yields

$$\Phi_A(t_0,t)=e_R^{-1}(t,t_0)L^{-1}(t)=e_R(t_0,t)L^{-1}(t)$$

which proves the claim

$$\Phi_A(t,\tau)=L(t)e_R(t,\tau)L^{-1}(\tau).$$

We conclude by showing that $L(t)$ is p-periodic. By (6.4) and Lemma 6.1,

$$\begin{aligned}
L(t+p)&=\Phi_A(t+p,t_0)e_R^{-1}(t+p,t_0)\\
&=\Phi_A(t+p,t_0+p)\Phi_A(t_0+p,t_0)e_R(t_0,t+p)\\
&=\Phi_A(t+p,t_0+p)\Phi_A(t_0+p,t_0)e_R(t_0,t_0+p)e_R(t_0+p,t+p)\\
&=\Phi_A(t+p,t_0+p)\Phi_A(t_0+p,t_0)e_R^{-1}(t_0+p,t_0)e_R(t_0+p,t+p)\\
&=\Phi_A(t+p,t_0+p)e_R^{-1}(t+p,t_0+p)\\
&=\Phi_A(t+p,t_0+p)e_R^{-1}(t,t_0).
\end{aligned}$$

Letting $t'=t+p$, we see that $\Phi_A(t',t_0+p)$ is a solution to the matrix dynamic equation

$$\Phi_A^{\Delta_{t'}}(t',t_0+p)=A(t')\Phi_A(t',t_0+p)=A(t+p)\Phi_A(t+p,t+0+p)=A(t)\Phi_A(t+p,t_0+p)$$

with initial conditions

$$\Phi_A(t', t_0 + p)|_{t'=t_0+p} = \Phi_A(t + p, t_0 + p)|_{t=t_0} = \Phi_A(t_0 + p, t_0 + p) = I.$$

But now $\Phi_A(t, t_0)$ is another solution to the same matrix dynamic initial value problem. Since the solutions to initial value problems are unique, we have

$$\Phi_A(t + p, t_0 + p) = \Phi_A(t, t_0).$$

Thus,

$$L(t + p) = \Phi_A(t + p, t_0 + p)e_R^{-1}(t, t_0) = \Phi_A(t, t_0)e_R^{-1}(t, t_0) = L(t).$$

□

In the following theorem we show that given any p-periodic nonautonomous system as in (6.1), we can create a corresponding autonomous system via the Floquet decomposition of the transition matrix for the nonautonomous system, which we have shown preserves the stability characteristics.

Theorem 6.2. *Let* $\Phi_A(t, t_0) = L(t)e_R(t, t_0)$ *as in Theorem 6.1. Then* $x(t) = \Phi_A(t, t_0)x_0$ *is a solution of the* p*-periodic nonautonomous system* (6.1) *if and only if* $z(t) = L^{-1}(t)x(t)$ *is a solution of the autonomous system*

$$z^{\Delta}(t) = R\, z(t), \quad z(t_0) = x_0. \tag{6.5}$$

Proof. Suppose that $x(t)$ is a solution to (6.1). Then

$$x(t) = \Phi_A(t, t_0)x_0 = L(t)e_R(t, t_0)x_0.$$

If we define

$$z(t) = L^{-1}(t)x(t) = L^{-1}(t)L(t)e_R(t, t_0)x_0 = e_R(t, t_0)x_0,$$

then it follows that $z(t)$ is a solution of (6.5).

Now suppose that $z(t) = L^{-1}(t)x(t)$ is a solution of the autonomous system (6.5). By [6], the solution is $z(t) = e_R(t, t_0)x_0$. By definition of $z(t)$ we have $x(t) = L(t)z(t)$. It follows that

$$x(t) = L(t)e_R(t, t_0)x_0 = \Phi_A(t, t_0)x_0,$$

so $x(t)$ is a solution of (6.1). □

We now give conditions on the transition matrix of the p-periodic nonautonomous system (6.1) and the corresponding solution to the autonomous system that guarantees the existence of a periodic solution to (6.1).

Theorem 6.3. *Given any $t_0 \in \mathbb{T}$, there exists an initial state $x(t_0) = x_0 \neq 0$ such that the solution of* (6.1) *is p-periodic if and only if at least one of the eigenvalues of $e_R(t_0 + p, t_0) = \Phi_A(t_0 + p, t_0)$ is* 1.

Proof. Suppose that given an initial time t_0 with $x(t_0) = x_0 \neq 0$, the solution $x(t)$ is p-periodic. By Theorem 6.1, there exists a Floquet decomposition of x given by

$$x(t) = \Phi_A(t, t_0)x_0 = L(t)e_R(t, t_0)L^{-1}(t_0)x_0.$$

Furthermore,

$$x(t + p) = L(t + p)e_R(t + p, t_0)L^{-1}(t_0)x_0 = L(t)e_R(t + p, t_0)L^{-1}(t_0)x_0.$$

Since $x(t) = x(t + p)$ and $L(t) = L(t + p)$ for each $t \in \mathbb{T}$, we have

$$e_R(t, t_0)L^{-1}(t_0)x_0 = e_R(t + p, t_0)L^{-1}(t_0)x_0,$$

which implies

$$e_R(t, t_0)L^{-1}(t_0)x_0 = e_R(t + p, t_0 + p)e_R(t_0 + p, t_0)L^{-1}(t_0)x_0.$$

Since $e_R(t + p, t_0 + p) = e_R(t, t_0)$,

$$e_R(t, t_0)L^{-1}(t_0)x_0 = e_R(t, t_0)e_R(t_0 + p, t_0)L^{-1}(t_0)x_0,$$

and thus

$$L^{-1}(t_0)x_0 = e_R(t_0+p,t_0)L^{-1}(t_0)x_0.$$

Since $L^{-1}(t_0)x_0 \neq 0$, we see that $L^{-1}(t_0)x_0$ is an eigenvector of the matrix $e_R(t_0+p,t_0)$ corresponding to an eigenvalue of 1.

Now suppose 1 is an eigenvalue of $e_R(t_0+p,t_0)$ with corresponding eigenvector z_0. Then z_0 is real-valued and nonzero. For any $t_0 \in \mathbb{T}$, $z(t) = e_R(t,t_0)z_0$ is p-periodic. Since 1 is an eigenvalue of $e_R(t_0+p,t_0)$ with corresponding eigenvector z_0 and $e_R(t+p,t_0+p) = e_R(t,t_0)$,

$$\begin{aligned} z(t+p) &= e_R(t+p,t_0)z_0 \\ &= e_R(t+p,t_0+p)e_R(t_0+p,t_0)z_0 \\ &= e_R(t+p,t_0+p)z_0 \\ &= e_R(t,t_0)z_0 \\ &= z(t). \end{aligned}$$

Using the Floquet decomposition from Theorem 6.1 and setting $x_0 = L(t_0)z_0$, we obtain the nontrivial solution of (6.1). Then

$$x(t) = \Phi_A(t,t_0)x_0 = L(t)e_R(t,t_0)L^{-1}(t_0)x_0 = L(t)e_R(t,t_0)z_0 = L(t)z(t),$$

which is p-periodic since $L(t)$ and $z(t)$ are p-periodic. □

6.2 The Nonhomogeneous Equation

We now consider the nonhomogeneous regressive time varying linear dynamic initial value problem

$$x^{\Delta}(t) = A(t)x(t) + f(t), \qquad x(t_0) = x_0, \tag{6.6}$$

where $A(t) \in \mathcal{R}(\mathbb{T},\mathbb{R}^{n\times n})$, $f(t) \in \mathrm{C}_{\mathrm{prd}}(\mathbb{T},\mathbb{R}^n) \cap \mathcal{R}(\mathbb{T},\mathbb{R}^n)$ and both are p-periodic for all $t \in \mathbb{T}$.

Lemma 6.2. *A solution $x(t)$ of equation* (6.6) *is p-periodic if and only if $x(t_0+p) = x(t_0)$.*

Proof. Suppose that $x(t)$ is p-periodic. Then by definition of a periodic function, $x(t_0+p) = x(t_0)$.

Now suppose that there exists a solution of (6.6) such that $x(t_0+p) = x(t_0)$. Define $z(t) = x(t+p) - x(t)$. By assumption and construction of $z(t)$, we have $z(t_0) = 0$. Furthermore,

$$\begin{aligned} z^{\Delta}(t) &= [A(t+p)x(t+p) + f(t+p)] - [A(t)x(t) + f(t)] \\ &= A(t)\left[x(t+p) - x(t)\right] \\ &= A(t)z(t). \end{aligned}$$

By uniqueness of solutions, we see that $z(t) \equiv 0$ for all $t \in \mathbb{T}$. Thus, $x(t) = x(t+p)$ for all $t \in \mathbb{T}$. □

The next theorem uses Lemma 6.2 to develop criteria for the existence of p-periodic solutions for any p-periodic vector-valued function $f(t)$.

Theorem 6.4. *For all $t_0 \in \mathbb{T}$ and for all p-periodic $f(t)$, there exists an initial state $x(t_0) = x_0$ such that the solution of* (6.6) *is p-periodic if and only if there does not exist a nonzero $z(t_0) = z_0$ with $t_0 \in \mathbb{T}$ such that the p-periodic homogeneous initial value problem*

$$z^{\Delta}(t) = A(t)z(t), \quad z(t_0) = z_0, \tag{6.7}$$

has a p-periodic solution.

Proof. For any t_0, $x(t_0) = x_0$ and p-periodic vector-valued function $f(t)$, by Theorem 2.7 the solution of (6.6) is

$$x(t) = \Phi_A(t,t_0)x_0 + \int_{t_0}^{t} \Phi_A(t,\sigma(\tau))f(\tau)\Delta\tau.$$

By Lemma 6.2, $x(t)$ is p-periodic if and only if $x(t_0) = x(t_0 + p)$ which is equivalent to

$$[I - \Phi_A(t_0 + p, t_0)]\, x_0 = \int_{t_0}^{t_0+p} \Phi_A(t_0 + p, \sigma(\tau)) f(\tau) \Delta\tau. \tag{6.8}$$

By Theorem 6.3, we must show the algebraic equation (6.8) has a solution for x_0 given any t_0 and any p-periodic $f(t)$ if and only if $e_R(t_0 + p, t_0)$ has no eigenvalues equal to one.

Let $e_R(\tau + p, \tau) = \Phi_A(\tau + p, \tau)$ for some $\tau \in \mathbb{T}$, and suppose that there are no eigenvalues equal to one. This is equivalent to

$$\det\, [I - \Phi_A(\tau + p, \tau)] \neq 0. \tag{6.9}$$

Since Φ_A is invertible, (6.9) is equivalent to

$$\begin{aligned} 0 &\neq \det\, [\Phi_A(t_0 + p, \tau + p)\, (I - \Phi_A(\tau + p, \tau))\, \Phi_A(\tau, t_0)] \\ &= \det\, [\Phi_A(t_0 + p, \tau + p)\Phi_A(\tau, t_0) - \Phi_A(t_0 + p, t_0)]\,. \end{aligned}$$

Since $\Phi_A(t_0 + p, \tau + p) = \Phi_A(t_0, \tau)$, as shown is Theorem 6.1, (6.9) is equivalent to the invertibility of $[I - \Phi_A(\tau + p, \tau)]$. Thus, (6.8) has a solution x_0 for any t_0 and for any p-periodic $f(t)$.

Now suppose that (6.8) has a solution for every t_0 and every p-periodic $f(t)$. Given an arbitrary $t_0 \in \mathbb{T}$, corresponding to any $n \times 1$ vector f_0, we define a regressive p-periodic vector valued function $f(t) \in \mathrm{C}_{\mathrm{prd}}(\mathbb{T}, \mathbb{R}^n)$ by

$$f(t) = \Phi_A(\sigma(t), t_0 + p) f_0, \quad t \in [t_0, t_0 + p)_{\mathbb{T}}\,, \tag{6.10}$$

extending this to the entire time scale $\mathbb{T}$ using the periodicity.

By construction of $f(t)$, we have

$$\int_{t_0}^{t_0+p} \Phi_A(t_0 + p, \sigma(\tau)) f(\tau) \Delta\tau = \int_{t_0}^{t_0+p} f_0 \Delta\tau = p f_0.$$

Thus, (6.8) becomes

$$[I - \Phi_A(t_0 + p, t_0)]\, x_0 = p f_0. \tag{6.11}$$

For any vector-valued function $f(t)$ that is constructed as in (6.10) and thus for any corresponding f_0, (6.11) has a solution for x_0 by assumption. Therefore,

$$\det\left[I - \Phi_A(t_0 + p, t_0)\right] \neq 0,$$

which is equivalent to (6.9). Thus, $e_R(t_0+p, t_0) = \Phi_A(t_0+p, t_0)$ has no eigenvalue of equal to 1. By Theorem 6.3, (6.7) has no periodic solution. □

6.3 Examples

6.3.1 Discrete Time Example

Consider the time scale $\mathbb{T} = \mathbb{Z}$ and the regressive (on $\mathbb{Z}$) time varying matrix

$$A(t) = \begin{bmatrix} -1 & \frac{2+(-1)^t}{2} \\ \frac{2+(-1)^t}{2} & -1 \end{bmatrix}$$

which have periods of 1 and 2, respectively. It can be verified that the transition matrix for the homogeneous periodic discrete linear system of difference equations

$$\Delta X(t) = \begin{bmatrix} -1 & \frac{2+(-1)^t}{2} \\ \frac{2+(-1)^t}{2} & -1 \end{bmatrix} X(t)$$

is given by

$$\Phi_A(t,0) = \frac{1}{2^{t+1}} \begin{bmatrix} (\sqrt{3})^t + (-\sqrt{3})^t & (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} \\ (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} & (\sqrt{3})^t + (-\sqrt{3})^t \end{bmatrix}.$$

A matrix R', as in Theorem 6.1, that satisfies the equation

$$e_{R'}(2,0) = \Phi_A(2,0) = \frac{1}{2^3} \begin{bmatrix} (\sqrt{3})^2 + (-\sqrt{3})^2 & (\sqrt{3})^3 + (-\sqrt{3})^3 \\ (\sqrt{3})^3 + (-\sqrt{3})^3 & (\sqrt{3})^2 + (-\sqrt{3})^2 \end{bmatrix}$$

which simplifies to

$$e_{R'}(2,0) = (I + R')^2 = \frac{1}{8} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$$

is

$$R' = \begin{bmatrix} \frac{\sqrt{3}}{2} - 1 & 0 \\ 0 & \frac{\sqrt{3}}{2} - 1 \end{bmatrix}.$$

Again, by Theorem 6.1, the 2-periodic matrix $L'(t)$ is given by

$$\begin{aligned} L'(t) &= \Phi_A(t,0)e_{R'}^{-1}(t,0) \\ &= \Phi_A(t,0)(I + R')^{-t} \\ &= \frac{1}{2^{t+1}} \begin{bmatrix} (\sqrt{3})^t + (-\sqrt{3})^t & (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} \\ (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} & (\sqrt{3})^t + (-\sqrt{3})^t \end{bmatrix} \begin{bmatrix} \left(\frac{\sqrt{3}}{2}\right)^{-t} & 0 \\ 0 & \left(\frac{\sqrt{3}}{2}\right)^{-t} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + (-1)^t & \sqrt{3} + (-1)^t(-\sqrt{3}) \\ \sqrt{3} + (-1)^t(-\sqrt{3}) & 1 + (-1)^t \end{bmatrix}. \end{aligned}$$

This examples illustrates how the unified Floquet theorem handles the case of a completely isolated time scale $\mathbb{T} = \mathbb{Z}$. In this case, since the time scale has constant graininess, the matrix R is computed by taking the pth root of the matrix $\Phi_A(t_0 + p, t_0)$ and subtracting the identity matrix, i.e. $R = (\Phi_A(2,0)^{\frac{1}{2}} - I)$. For any time scale with constant positive graininess h and period p,

$$R = \frac{1}{h}(\Phi_A(t_0 + p, t_0)^{\frac{h}{p}} - I).$$

6.3.2 Continuous Time Example

Consider the time scale $\mathbb{T} = \mathbb{R}$ and the time varying matrix

$$A(t) = \begin{bmatrix} -1 & 0 \\ \sin(t) & 0 \end{bmatrix}$$

which has a period of 2π. It can be verified that the transition matrix for the homogeneous periodic linear system of differential equations

$$\dot{X}(t) = \begin{bmatrix} -1 & 0 \\ \sin(t) & 0 \end{bmatrix} X(t)$$

is given by

$$\Phi_A(t,0) = \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2} - \frac{e^{-t}(\cos(t)+\sin(t))}{2} & 1 \end{bmatrix}.$$

A matrix R', as in Theorem 6.1, that satisfies the equation

$$e_{R'}(2\pi,0) = e^{2\pi R'} = \Phi_A(2\pi,0) = \begin{bmatrix} e^{-2\pi} & 0 \\ \frac{1}{2} - \frac{e^{-2\pi}}{2} & 1 \end{bmatrix}$$

is

$$R' = \frac{1}{2\pi}\ln \Phi_A(2\pi,0) = \begin{bmatrix} -1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

We can now conclude that

$$e^{R't} = \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2} - \frac{e^{-t}}{2} & 1 \end{bmatrix} \quad \text{and thus} \quad e^{-R't} = \begin{bmatrix} e^{t} & 0 \\ \frac{1}{2} - \frac{e^{t}}{2} & 1 \end{bmatrix}.$$

Again, by Theorem 6.1, the 2π-periodic matrix $L'(t)$ is given by

$$\begin{aligned} L'(t) &= \Phi_A(t,0)e_{R'}^{-1}(t,0) \\ &= \Phi_A(t,0)e^{-R't} \\ &= \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2} - \frac{e^{-t}(\cos(t)+\sin(t))}{2} & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ \frac{1}{2} - \frac{e^{t}}{2} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \frac{1}{2} - \frac{\cos(t)+\sin(t)}{2} & 1 \end{bmatrix}. \end{aligned}$$

This examples demonstrates how the unified Floquet theorem handles the case of the time scale $\mathbb{T} = \mathbb{R}$. In this case, since the time scale has a constant graininess of $\mu(t) \equiv 0$, the matrix R is computed by taking the natural logarithm of the matrix $\Phi_A(2\pi, 0)$ and multiplying by $\frac{1}{2\pi}$, i.e. $R = \frac{1}{2\pi}\ln\Phi_A(2\pi,0)$. In general, for the time scale $\mathbb{T} = \mathbb{R}$, and matrix $A(t)$ with period p,

$$R = \frac{1}{p}\ln \Phi_A(t_0 + p, t_0).$$

6.3.3 Time Scale Example

We start by stating a lemma from [6]. It will be used in finding the matrix R in the following example.

Lemma 6.3. *The initial value problem*

$$y^{\Delta}(t) = \lambda_1(t)y(t) + e_{\lambda_2(t)}(t, t_0), \quad y(t_0) = 0 \tag{6.12}$$

has the solution

$$y(t) = \int_{t_0}^{t} e_{\lambda_1(t)}(t, \sigma(\tau))e_{\lambda_2(\tau)}(\tau, t_0)\Delta\tau. \tag{6.13}$$

Consider the time scale $\mathbb{T} = \mathbb{P}_{1,1}$ and the regressive (on $\mathbb{P}_{1,1}$) time varying matrix

$$A(t) = \begin{bmatrix} -3 + \sin(2\pi t) & 1 \\ 0 & -3 \end{bmatrix}$$

which have periods of 2 and 1, respectively. Note that it is correct to say the matrix $A(t)$ is 2-periodic as well. It is tedious but straightforward to verify that the transition matrix for the homogeneous periodic dynamic linear system

$$X^{\Delta}(t) = \begin{bmatrix} -3 + \sin(2\pi t) & 1 \\ 0 & -3 \end{bmatrix} X(t) \tag{6.14}$$

is given by

$$\Phi_A(t, 0) = \begin{bmatrix} e_{-3+\sin(2\pi\tau)}(t, 0) & \int_0^t e_{-3+\sin(2\pi\tau)}(t, \sigma(\tau))e_{-3}(\tau, 0)\Delta\tau \\ 0 & e_{-3}(t, 0) \end{bmatrix}. \tag{6.15}$$

Following the Putzer Algorithm in [6], we see that the matrix R' that satisfies

$$e_{R'}(2, 0) = \Phi_A(2, 0) = \begin{bmatrix} -2e^{-3} & \int_0^2 e_{-3+\sin(2\pi t)}(t, \sigma(\tau))e_{-3}(\tau, 0)\Delta\tau \\ 0 & -2e^{-3} \end{bmatrix}$$

as in Theorem 6.1 is

$$R' = \begin{bmatrix} -3 & C \\ 0 & -3 \end{bmatrix},$$

where $C = -e^3 \int_0^2 e_{-3+\sin(2\pi\tau)}(2,\sigma(s))e_{-3}(s,0)\Delta s$ and

$$e_{-3+\sin(2\pi\tau)}(2,\sigma(s)) := \exp\left(\int_{\sigma(s)}^{2} \frac{1}{\mu(\tau)} \operatorname{Log}(1+\mu(\tau)(-3+\sin(2\pi\tau)))\Delta\tau\right),$$

which is why we have $\sin(2\pi\tau)$ in the integral. Since $e_{R'}(2,0) = \Phi_A(2,0)$, we see

$$e_{R'}(t,0) = \begin{bmatrix} e_{-3}(t,0) & C\int_0^t e_{-3}(t,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & e_{-3}(t,0) \end{bmatrix},$$

and consequently

$$e_{R'}^{-1}(t,0) = \frac{1}{(e_{-3}(t,0))^2} \begin{bmatrix} e_{-3}(t,0) & -C\int_0^t e_{-3}(t,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & e_{-3}(t,0) \end{bmatrix}.$$

Again, by Theorem 6.1, the 2-periodic matrix $L'(t)$ is given by

$$\begin{aligned} L'(t) &= \Phi_A(t,0)e_{R'}^{-1}(t,0) \\ &= \frac{1}{(e_{-3}(t,0))^2} \begin{bmatrix} e_{-3+\sin(2\pi t)}(t,0) & \int_0^t e_{-3+\sin(2\pi\tau)}(t,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot \\ &\quad \begin{bmatrix} e_{-3}(t,0) & -C\int_0^t e_{-3}(t,\sigma(\tau))e_{-3}(\tau,0)\Delta\tau \\ 0 & e_{-3}(t,0) \end{bmatrix}, \end{aligned}$$

which is obviously 2-periodic on $\mathbb{P}_{1,1}$. Thus, the Floquet decomposition of the transition matrix is seen to be $\Phi_A(t,0) = L'(t)e_{R'}(t,0)$.

The unified Floquet theorem can also be used on a time scale with nonconstant graininess. In this case, since the time scale has a nonconstant graininess, the matrix R is computed using the Putzer Algorithm. There does not currently exist a closed form for the matrix R when working on a time scale with nonconstant graininess.

CHAPTER SEVEN

Floquet Multipliers, Floquet Exponents, and a Spectral Mapping Theorem

Suppose that $\Phi_A(t, t_0)$ is the transition matrix and $\Phi(t)$ is the fundamental matrix at $t = \tau$ (i.e. $\Phi(\tau) = I$) for the system (6.1). Then we can write any fundamental matrix $\Psi(t)$ as

$$\Psi(t) = \Phi(t)\Psi(\tau) \quad \text{or} \quad \Psi(t) = \Phi_A(t, t_0)\Psi(t_0).$$

Definition 7.1. Let $x_0 \in \mathbb{R}^n$ be a nonzero vector and $\Psi(t)$ be any fundamental matrix for the system (6.1). The vector solution of the system with initial condition $x(t_0) = x_0$ is given by $x(t) = \Phi_A(t, t_0)x_0 = \Psi(t)\Psi^{-1}(t_0)x_0$. The operator $M : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$M(x_0) = \Phi_A(t_0 + p, t_0)x_0 = \Psi(t_0 + p)\Psi^{-1}(t_0)x_0$$

is called a *monodromy operator*. The eigenvalues of the monodromy operator are called the *Floquet* (or *characteristic*) *multipliers* of the system (6.1).

The following theorem establishes that characteristic multipliers are nonzero complex numbers intrinsic to the periodic system—they do not depend on the choice of the fundamental matrix. This a generalization of the result dealing with the eigenvalues and invertibility of monodromy operators in [10].

Theorem 7.1. *The following statements are valid for the system* (6.1).

(1) *Every monodromy operator is invertible. In particular, every characteristic multiplier is nonzero.*

(2) *If M_1 and M_2 are monodromy operators, then they have the same eigenvalues. In particular, there are exactly n characteristic multipliers, counting multiplicities.*

Proof. To prove (1), observe that by the definition of the monodromy operator, it is invertible for all $t \in \mathbb{T}$ with $t \geq t_0$.

To prove (2), we develop a property for fundamental matrices from Theorem 6.3. Let $\Psi_1(t)$ be a fundamental matrix for the p-periodic system (6.1) at $t = \tau$. Define $\Upsilon(t) := \Psi_1(t+p)$ and $C := \Psi_1^{-1}(\tau)\Psi_1(\tau+p)$. Then $\Upsilon(\tau) = \Psi_1(\tau+p) = \Psi_1(\tau)C$ and by uniqueness of solutions, $\Upsilon(t) = \Psi_1(t+p) = \Psi_1(t)C$. The property that follows is

$$\Psi_1(t+p) = \Psi_1(t)C = \Psi_1(t)\Psi_1^{-1}(\tau)\Psi_1(\tau+p).$$

If $\Psi_2(t)$ is another fundamental matrix, then $\Psi_2(t) = \Psi_1(t)\Psi_2(\tau)$ and $\Psi_2(t) = \Phi_A(t,t_0)\Psi_2(t_0)$.

Consider the monodromy operator given by $M(x_0) = \Psi_2(t_0+p)\Psi_2^{-1}(t_0)x_0$, and note that

$$\begin{aligned}
\Psi_2(t_0+p)\Psi_2^{-1}(t_0) &= \Psi_1(t_0+p)\Psi_2(\tau)\Psi_2^{-1}(\tau)\Psi_1^{-1}(t_0) \\
&= \Psi_1(t_0+p)\Psi_1^{-1}(t_0) \\
&= \Psi_1(t_0)\Psi_1^{-1}(\tau)\Psi_1(\tau+p)\Psi_1^{-1}(t_0) \\
&= \Psi_1(t_0)\Psi_1(\tau+p)\Psi_1^{-1}(t_0),
\end{aligned}$$

or in terms of the transition matrix $\Phi_A(t,t_0)$,

$$\Psi_2(t_0+p)\Psi_2^{-1}(t_0) = \Phi_A(t_0+p,t_0)\Psi_2(t_0)\Psi_2^{-1}(t_0)\Phi_A^{-1}(t_0,t_0) = \Phi_A(t_0+p,t_0).$$

□

In particular, the eigenvalues of the operator $\Psi_1(\tau+p)$ are the same as the eigenvalues of the monodromy operator M. Similarly, in terms of the transition matrix, the eigenvalues of $\Phi_A(t_0+p,t_0)$ are the same as the eigenvalues of the monodromy operator M. Thus, all monodromy operators have the same eigenvalues.

With the Floquet normal form $\Phi_A(t,t_0) = \Psi_1(t)\Psi_1^{-1}(t_0) = L(t)e_R(t,t_0)L^{-1}(t_0)$ of the transition matrix for the system (6.1) one on hand, and the monodromy operator representation

$$M(x_0) = \Phi_A(t_0 + p, t_0)x_0 = \Psi_1(t_0 + p)\Psi_1^{-1}(t_0)x_0$$

on the other, together we conclude

$$\Phi_A(t_0 + p, t_0) = \Psi_1(t_0 + p)\Psi_1^{-1}(t_0) = L(t_0)e_R(t_0 + p, t_0)L^{-1}(t_0).$$

Thus, the characteristic multipliers of the system are the eigenvalues of the matrix $e_R(t_0 + p, t_0)$. The number $\gamma \in \mathbb{C}$ is a *Floquet* (or *characteristic*) *exponent* of the p-periodic system (6.1) if λ is a Floquet multiplier and $e_\gamma(t_0 + p, t_0) = \lambda$.

Lemma 7.1. *Let A be an $n \times n$ constant matrix and T be any nonsingular $n \times n$ matrix. Then $e_{TAT^{-1}}(t, t_0) = Te_A(t, t_0)T^{-1}$.*

Proof. Following the Putzer Algorithm, suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then

$$e_A(t, t_0) = \sum_{i=0}^{n-1} r_{i+1}(t)P_i,$$

where $r(t) := (r_1(t), r_2(t), \ldots, r_n(t))$ is the solution of the IVP

$$r^\Delta(t) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \ddots & 0 \\ 0 & 1 & \lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix}, \quad r(t_0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and the P-matrices $P_0, P_1, \ldots, P_n$ are recursively defined by $P_0 = I$ and

$$P_{k+1} = (A - \lambda_{k+1}I)P_k \quad \text{for} \quad 0 \le k \le n-1.$$

Since the matrices A and TAT^{-1} have the same eigenvalues, the corresponding (scalar) functions $r_i(t)$ are identical.

Suppose

$$e_A(t,t_0) = \sum_{i=0}^{n-1} r_{i+1}(t)P_i, \quad \text{and} \quad e_{TAT^{-1}}(t,t_0) = \sum_{i=0}^{n-1} r_{i+1}(t)Q_i.$$

To conclude the proof, we show that $TP_{k+1}T^{-1} = Q_{k+1}$ for all $0 \le k \le n-1$. For any $0 \le k \le n-1$,

$$\begin{aligned}
TP_{k+1}T^{-1} &= T(A-\lambda_{k+1}I)(A-\lambda_k I)\cdots(A-\lambda_1 I)T^{-1} \\
&= T(A-\lambda_{k+1}I)T^{-1}T(A-\lambda_k I)T^{-1}\cdots T(A-\lambda_1 I)T^{-1} \\
&= (TA-\lambda_{k+1}T)T^{-1}T(A-\lambda_k I)T^{-1}\cdots T(At^{-1}-\lambda_1 T^{-1}) \\
&= (TAT^{-1}-\lambda_{k+1}I)(TAT^{-1}-\lambda_k I)\cdots(TAT^{-1}-\lambda_1 I) \\
&= Q_{k+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
Te_A(t,t_0)T^{-1} &= T\left(\sum_{i=0}^{n-1} r_{i+1}(t)P_i\right)T^{-1} \\
&= \sum_{i=0}^{n-1} r_{i+1}(t)TP_iT^{-1} \\
&= \sum_{i=0}^{n-1} r_{i+1}(t)Q_i \\
&= e_{TAT^{-1}}(t,t_0).
\end{aligned}$$

□

The next result is an interesting spectral mapping theorem for time scales. Let spec(A) denote the spectrum of A, that is, the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is singular. Then, for our finite dimensional matrix A, spec(A) coincides with the set of eigenvalues of A. The fact we obtain from Theorem 7.2 is that $e_{\text{spec}(A)} = \text{spec}(e_A)$.

Theorem 7.2 (Spectral Mapping Theorem for Time Scales). *Suppose that A is an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, repeated according to multiplicities. Then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k and the eigenvalues of e_A are $e_{\lambda_1}, \ldots, e_{\lambda_n}$.*

Proof. By induction for the dimension n, we start by noting that the theorem is valid for 1×1 matrices. Suppose that it is true for all $(n-1) \times (n-1)$ matrices. Take λ_1 and let $v \neq 0$ denote a corresponding eigenvector such that $Av = \lambda_1 v$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the usual basis of $\mathbb{C}^n$. There exists a nonsingular matrix S such that $Sv = \mathbf{e}_1$. Thus we have $SAS^{-1}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1$, and the matrix SAS^{-1} has the block form

$$SAS^{-1} = \begin{bmatrix} \lambda_1 & * \\ 0 & \widetilde{A} \end{bmatrix}.$$

The matrix SA^kS^{-1} has the same block form, only with block diagonal elements λ_1^k and $\widetilde{A}^k$. Clearly, the eigenvalues of this block matrix are λ_1^k together with the eigenvalues of $\widetilde{A}^k$. By induction, the eigenvalues of $\widetilde{A}^k$ are the kth powers of the eigenvalues of $\widetilde{A}$. This proves the second statement of the theorem.

We note that the structure of each P_i in Lemma 7.1 depends explicitly on the two matrices A and I. Since we chose the matrix S so that SAS^{-1} is block diagonal, by construction, the matrix $e_{SAS^{-1}}$ is also block diagonal, with block diagonal elements e_{λ_1} and $e_{\tilde{A}}$, which can be verified by the Putzer Algorithm and time scale integration by parts. Using induction, it follows that the eigenvalues of $e_{\tilde{A}}$ are $e_{\lambda_2}, \ldots, e_{\lambda_n}$. Thus, the eigenvalues of $e_{SAS^{-1}} = Se_AS^{-1}$ are $e_{\lambda_1}, \ldots, e_{\lambda_n}$. □

We know that the eigenvalues of the matrix $e_R(t_0+p, t_0)$ are the Floquet multipliers. Theorem 7.2 also helps us answer affirmatively the question of whether or not the eigenvalues of the matrix R in the Floquet decomposition $\Phi_A(t, t_0) = L(t)e_R(t, t_0)$ are Floquet exponents. However, in Theorem 7.3, we will see that although the Floquet exponents are the eigenvalues of the matrix R, they are not unique.

We first introduce the definition of a Hilger purely imaginary number.

Definition 7.2. Let $-\frac{\pi}{h} < \omega \leq \frac{\pi}{h}$. The *Hilger purely imaginary number* $\overset{\circ}{\imath}\omega$ is defined by

$$\overset{\circ}{\imath}\omega = \frac{e^{i\omega h} - 1}{h}.$$

For $z \in \mathbb{C}_h$, we have that $\mathring{\iota}\mathrm{Im}_h(z) \in \mathbb{I}_h$. Also, when $h = 0$, $\mathring{\iota}\omega = i\omega$.

Theorem 7.3 (Nonuniqueness of Floquet Exponents). *Suppose that $\gamma \in \mathcal{R}$ is a (possibly complex) Floquet exponent, λ is the corresponding characteristic multiplier of the p-periodic Floquet system (6.1) such that $e_\gamma(t_0+p, t_0) = \lambda$, and $\mathbb{T}$ is a p-periodic time scale. Then $\gamma \oplus \mathring{\iota}\frac{2\pi k}{p}$ is also a Floquet exponent for all $k \in \mathbb{Z}$.*

Proof. Observe that for any $k \in \mathbb{Z}$ and $t_0 \in \mathbb{T}$,

$$
\begin{aligned}
e_{\gamma \oplus \mathring{\iota}\frac{2\pi k}{p}}(t_0+p, t_0) &= e_\gamma(t_0+p, t_0) e_{\mathring{\iota}\frac{2\pi k}{p}}(t_0+p, t_0) \\
&= e_\gamma(t_0+p, t_0) \exp\left(\int_{t_0}^{t_0+p} \frac{\mathrm{Log}(1+\mu(\tau)\mathring{\iota}\frac{2\pi k}{p})}{\mu(\tau)} \Delta\tau \right) \\
&= e_\gamma(t_0+p, t_0) \exp\left(\int_{t_0}^{t_0+p} \frac{\mathrm{Log}(1+\mu(\tau)\frac{e^{i2\pi k\mu(\tau)/p}-1}{\mu(\tau)})}{\mu(\tau)} \Delta\tau \right) \\
&= e_\gamma(t_0+p, t_0) \exp\left(\int_{t_0}^{t_0+p} \frac{\mathrm{Log}(e^{i2\pi k\mu(\tau)/p})}{\mu(\tau)} \Delta\tau \right) \\
&= e_\gamma(t_0+p, t_0) \exp\left(\int_{t_0}^{t_0+p} \frac{i2\pi k\mu(\tau)/p}{\mu(\tau)} \Delta\tau \right) \\
&= e_\gamma(t_0+p, t_0) \exp\left(\int_{t_0}^{t_0+p} \frac{i2\pi k}{p} \Delta\tau \right) \\
&= e_\gamma(t_0+p, t_0) e^{i2\pi k} \\
&= e_\gamma(t_0+p, t_0).
\end{aligned}
$$

□

Lemma 7.2. *Let $\mathbb{T}$ be a p-periodic time scale and $k \in \mathbb{Z}$. Then functions $e_{\mathring{\iota}\frac{2\pi k}{p}}(t, t_0)$ and $e_{\ominus\mathring{\iota}\frac{2\pi k}{p}}(t, t_0)$ are periodic functions.*

Proof. Let $t \in \mathbb{T}$. Then

$$
e_{\mathring{\iota}\frac{2\pi k}{p}}(t+p, t_0) = e^{\frac{i2\pi k(t+p-t_0)}{p}} = e^{\frac{i2\pi k(t-t_0)}{p}} e^{\frac{i2\pi kp}{p}} = e^{\frac{i2\pi k(t-t_0)}{p}} = e_{\mathring{\iota}\frac{2\pi k}{p}}(t, t_0).
$$

Therefore, $e_{\mathring{\iota}\frac{2\pi k}{p}}(t, t_0)$ is a p-periodic function. The fact that $e_{\ominus\mathring{\iota}\frac{2\pi k}{p}}(t, t_0)$ is p-periodic follows easily from the relationship $e_{\ominus\mathring{\iota}\frac{2\pi k}{p}}(t, t_0) = e_{\mathring{\iota}\frac{2\pi k}{p}}(t_0, t)$. □

Lemma 7.3. *If γ is a characteristic exponent for the system* (6.1) *and $\Phi_A(t,t_0)$ is the transition matrix, then Φ_A has the Floquet decomposition $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ such that γ is an eigenvalue of R.*

Proof. Let $L'(t)e_{R'}(t,t_0)$ be a Floquet decomposition of $\Phi_A(t,t_0)$. By definition of the characteristic exponents, there is a characteristic multiplier λ such that $e_\gamma(p+t_0,t_0) = \lambda$, and, by Theorem 7.2, there is an eigenvalue of ν of R' such that $e_\nu(p+t_0,t_0) = \lambda$. Also, by Lemma 7.3, there is some integer k such that $\nu = \gamma \oplus \mathring{\imath}\frac{2\pi k}{p}$.

Define $R := R' \ominus \mathring{\imath}\frac{2\pi k}{p} I$ (which is equivalent to $R' := R \oplus \mathring{\imath}\frac{2\pi k}{p} I$ and in this case since $\mathring{\imath}\frac{2\pi k}{p} I$ is a diagonal matrix, $R' := \mathring{\imath}\frac{2\pi k}{p} I \oplus R$) and $L(t) := L'(t)e_{\mathring{\imath}\frac{2\pi k}{p} I}(t,t_0)$. Then γ is an eigenvalue of R, L is a p-periodic function, and

$$L(t)e_R(t,t_0) = L'(t)e_{\mathring{\imath}\frac{2\pi k}{p} I}(t,t_0)e_R(t,t_0) = L'(t)e_{\mathring{\imath}\frac{2\pi k}{p} I \oplus R}(t,t_0) = L'(t)e_{R'}(t,t_0).$$

It follows that $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ is another Floquet decomposition where γ is an eigenvalue of R. □

The following theorem is used to classify the possible types of solutions that can arise with periodic systems.

Theorem 7.4. *If λ is a characteristic multiplier of the p-periodic system* (6.1) *and $e_\gamma(t_0+p,t_0) = \lambda$ for some $t_0 \in \mathbb{T}$, then there exists a* (*possibly complex*) *nontrivial solution of the form*

$$x(t) = e_\gamma(t,t_0)q(t)$$

where q is a p-periodic function. Moreover, for this solution $x(t+p) = \lambda x(t)$.

Proof. Let $\Phi_A(t,t_0)$ be the transition matrix for (6.1). By Lemma 7.3, there is a Floquet decomposition $\Phi_A(t,t_0) = L(t)e_R(t,t_0)$ such that γ is an eigenvalue of R. So there exists a vector $v \neq 0$ such that $Rv = \gamma v$. It follows that $e_R(t,t_0)v = e_\gamma(t,t_0)v$, and therefore the solution $x(t) := \Phi_A(t,t_0)v$ can be represented in the form

$$x(t) = L(t)e_R(t,t_0)v = e_\gamma(t,t_0)L(t)v.$$

The solution required by the first statement of the theorem is obtained by defining $q(t) := L(t)v$. The second statement of the theorem is proved by the following:

$$\begin{aligned}
x(t+p) &= e_\gamma(t+p, t_0)q(t+p) \\
&= e_\gamma(t+p, t_0+p)e_\gamma(t_0+p, t_0)q(t) \\
&= e_\gamma(t_0+p, t_0)e_\gamma(t+p, t_0+p)q(t) \\
&= e_\gamma(t_0+p, t_0)e_\gamma(t, t_0)L(t)v \\
&= e_\gamma(t_0+p, t_0)x(t) \\
&= \lambda x(t).
\end{aligned}$$

□

Corollary 7.1. *Suppose that $\gamma_1, \ldots, \gamma_n$ are the eigenvalues of R in the Floquet decomposition $\Phi_A(t, t_0) = L(t)e_R(t, t_0)$. Then if $\gamma_1, \ldots, \gamma_n \in \mathcal{S}(\mathbb{T})$ and there exists a $\delta > 0$ such that $0 < \delta^{-1} \leq |1 + \mu(t)\gamma_i|$ for all $i = 1, \ldots, n$ and $t \in \mathbb{T}^\kappa$, then the system* (6.1) *is exponentially stable.*

Proof. Corollary 7.1 is true by [40, Thm. 5.1(b)]. □

The next corollary is motivated by [10, Thm. 2.53].

Corollary 7.2. *Suppose that $\lambda_1, \ldots, \lambda_n$ are the Floquet multipliers for the p-periodic system* (6.1).

(1) *If all the Floquet multipliers have modulus less than one, then the system* (6.1) *is exponentially stable.*

(2) *If all of the Floquet multipliers have modulus less than or equal to one, then the system* (6.1) *is stable.*

(3) *If at least one of the Floquet multipliers have modulus greater than one, then the system* (6.1) *is unstable.*

Proof. Parts (1), (2), and (3) are true by Definition 3.7 and Theorem 3.4. □

Theorem 7.5. *Suppose that* λ_1 *and* λ_2 *are characteristic multipliers of the p-periodic system* (6.1) *and* γ_1 *and* γ_2 *are Floquet exponents such that* $e_{\gamma_1}(t_0+p,t_0)=\lambda_1$ *and* $e_{\gamma_2}(t_0+p,t_0)=\lambda_2$. *If* $\lambda_1 \neq \lambda_2$, *then there are p-periodic functions* q_1 *and* q_2 *such that*

$$x_1(t)=e_{\gamma_1}(t,t_0)q_1(t) \quad \textit{and} \quad x_2(t)=e_{\gamma_2}(t,t_0)q_2(t)$$

are linearly independent solutions.

Proof. As in Lemma 7.3, let $\Phi_A(t,t_0)=L(t)e_R(t,t_0)$ be such that γ_1 is an eigenvalue of R with corresponding (nonzero) eigenvector v_1. Since λ_2 is an eigenvalue of the monodromy matrix $\Phi_A(t_0+p,t_0)$, by Theorem 7.2 there is an eigenvalue γ of R such that $e_\gamma(t_0+p,t_0)=\lambda_2=e_{\gamma_2}(t_0+p,t_0)$. Hence $\gamma_2=\gamma\oplus \mathring{\imath}\frac{2\pi k}{p}$ for some $k\in\mathbb{Z}$. Also, $\gamma\neq\gamma_1$ since $\lambda_1\neq\lambda_2$. Thus, if v_2 is a nonzero eigenvector of R corresponding to the eigenvalue γ, then the eigenvectors v_1 and v_2 are linearly independent.

As in the proof of Theorem 7.4, there are solutions of the form

$$x_1(t)=e_{\gamma_1}(t,t_0)L(t)v_1, \qquad x_2(t)=e_{\gamma}(t,t_0)L(t)v_2.$$

Because $x_1(t_0)=v_1$ and $x_2(t_0)=v_2$, these solutions are linearly independent. Finally, x_2 can be written as

$$x_2(t)=\left(e_{\gamma\oplus\mathring{\imath}\frac{2\pi k}{p}}(t,t_0)\right)\left(e_{\ominus\mathring{\imath}\frac{2\pi k}{p}}(t,t_0)L(t)v_2\right),$$

where $q_2(t):=e_{\ominus\mathring{\imath}\frac{2\pi k}{p}}(t,t_0)L(t)v_2$. $\square$

CHAPTER EIGHT

Examples Revisited

We now revisit the examples from Section 6.3 and show how the Floquet Theory from Chapter 7 can be applied.

8.1 Discrete Time Example

With the discrete time example, the Floquet exponent is $\frac{\sqrt{3}}{2}-1$. It can be shown that $\gamma=-\frac{\sqrt{3}}{2}-1$ is also a Floquet exponent. However, it is not an eigenvalue of the original matrix R'. Define $R := R' \ominus \mathring{\imath}\pi I$, as in Theorem 7.3, with $k=1$ and $p=2$. Thus,

$$\begin{aligned} R &= \begin{bmatrix} \frac{\sqrt{3}}{2}-1 & 0 \\ 0 & \frac{\sqrt{3}}{2}-1 \end{bmatrix} \ominus \begin{bmatrix} \mathring{\imath}\frac{2\pi}{2} & 0 \\ 0 & \mathring{\imath}\frac{2\pi}{2} \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{\sqrt{3}}{2}-1\right)\ominus\mathring{\imath}\pi & 0 \\ 0 & \left(\frac{\sqrt{3}}{2}-1\right)\ominus\mathring{\imath}\pi \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\sqrt{3}}{2}-1 & 0 \\ 0 & -\frac{\sqrt{3}}{2}-1 \end{bmatrix}. \end{aligned}$$

Then

$$e_R(t,0)=(I+R)^t=\begin{bmatrix} \left(-\frac{\sqrt{3}}{2}\right)^t & 0 \\ 0 & \left(-\frac{\sqrt{3}}{2}\right)^t \end{bmatrix} \text{ and } e_{\mathring{\imath}\pi I}(t,0)=(-I)^t=\begin{bmatrix} (-1)^t & 0 \\ 0 & (-1)^t \end{bmatrix}.$$

Using the original Lyapunov transformation matrix $L'(t)$ from the discrete time example, define

$$L(t) := L'(t)e_{\mathring{\imath}\frac{2\pi k}{2}I}(t,0) = L'(t)e_{\mathring{\imath}\pi I}(t,0),$$

and thereby obtain

$$L(t) = \frac{1}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}\begin{bmatrix} (-1)^t & 0 \\ 0 & (-1)^t \end{bmatrix}$$
$$= \frac{(-1)^t}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}.$$

Thus,

$$L'(t)e_{R'}(t,0) = L'(t)e_{\mathring{i}\pi I \oplus R}(t,0) = L'(t)e_{\mathring{i}\pi I}(t,0)e_R(t,0) = L(t)e_R(t,0),$$

and so

$$L'(t)e_{R'}(t,0) = \frac{1}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}\begin{bmatrix} \left(\frac{\sqrt{3}}{2}\right)^t & 0 \\ 0 & \left(\frac{\sqrt{3}}{2}\right)^t \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}\begin{bmatrix} (-1)^t & 0 \\ 0 & (-1)^t \end{bmatrix}$$
$$\cdot \begin{bmatrix} \left(-\frac{\sqrt{3}}{2}\right)^t & 0 \\ 0 & \left(-\frac{\sqrt{3}}{2}\right)^t \end{bmatrix}$$
$$= \frac{(-1)^t}{2}\begin{bmatrix} 1+(-1)^t & \sqrt{3}+(-1)^t(-\sqrt{3}) \\ \sqrt{3}+(-1)^t(-\sqrt{3}) & 1+(-1)^t \end{bmatrix}\begin{bmatrix} \left(-\frac{\sqrt{3}}{2}\right)^t & 0 \\ 0 & \left(-\frac{\sqrt{3}}{2}\right)^t \end{bmatrix}$$
$$= L(t)e_R(t,0).$$

Therefore, $\Phi_A(t,0) = L(t)e_R(t,0)$ is another Floquet decomposition of the transition matrix and $\gamma = (\frac{\sqrt{3}}{2}-1)\ominus\mathring{i}\pi = -\frac{\sqrt{3}}{2}-1$ is a Floquet exponent as well as an eigenvalue of R which corresponds to the Floquet multiplier $\lambda = \frac{3}{4}$; that is, $e_{(\frac{\sqrt{3}}{2}-1)\ominus\mathring{i}\pi}(2,0) = e_{(-\frac{\sqrt{3}}{2}-1)}(2,0) = \frac{3}{4}$.

8.2 Continuous Time Example

Looking at the continuous time example, the original matrix R' was found to be

$$R' = \frac{1}{2\pi} \ln \Phi_A(2\pi, 0) = \begin{bmatrix} -1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Again define $R := R' \ominus \mathring{\imath}\frac{2\pi k}{2\pi} I = R' \ominus \mathring{\imath} I = R' - iI$ as in Theorem 7.3, with $k = 1$, $p = 2\pi$, and $\mu(t) \equiv 0$. Thus,

$$R = \begin{bmatrix} -1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1-i & 0 \\ \frac{1}{2} & -i \end{bmatrix}.$$

Hence

$$e^{Rt} = \begin{bmatrix} e^{(-1-i)t} & 0 \\ \frac{e^{-it}}{2} - \frac{e^{(-1-i)t}}{2} & e^{-it} \end{bmatrix},$$

and

$$\Phi_A(2\pi, 0) = e^{2\pi R} = \begin{bmatrix} e^{-2\pi} & 0 \\ \frac{1}{2} - \frac{e^{-2\pi}}{2} & 1 \end{bmatrix}.$$

Using the original Lyapunov transformation matrix $L'(t)$ from the continuous time example, we define

$$L(t) := L'(t) e_{\mathring{\imath}\frac{2\pi k}{2\pi} I}(t, 0) = L'(t) e^{iIt},$$

so that

$$L(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} - \frac{\cos(t)+\sin(t)}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{it} & 0 \\ 0 & e^{it} \end{bmatrix} = \begin{bmatrix} e^{it} & 0 \\ \frac{e^{it}}{2} - \frac{e^{it}(\cos(t)+\sin(t))}{2} & e^{it} \end{bmatrix}$$

and thus

$$L'(t) e^{R't} = L'(t) e^{(iI+R)t} = L'(t) e^{iIt} e^{Rt} = L(t) e^{Rt}.$$

Now we see

$$L'(t)e^{R't} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} - \frac{\cos(t)+\sin(t)}{2} & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2} - \frac{e^{-t}}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{1}{2} - \frac{\cos(t)+\sin(t)}{2} & 1 \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{it} \end{bmatrix} \begin{bmatrix} e^{(-1-i)t} & 0 \\ \frac{e^{-it}}{2} - \frac{e^{(-1-i)t}}{2} & e^{-it} \end{bmatrix}$$

$$= \begin{bmatrix} e^{it} & 0 \\ \frac{e^{it}}{2} - \frac{e^{it}(\cos(t)+\sin(t))}{2} & e^{it} \end{bmatrix} \begin{bmatrix} e^{(-1-i)t} & 0 \\ \frac{e^{-it}}{2} - \frac{e^{(-1-i)t}}{2} & e^{-it} \end{bmatrix}$$

$$= L(t)e^{Rt}.$$

Therefore, $\Phi_A(t,0) = L(t)e_R(t,0)$ is another Floquet decomposition of the transition matrix and $\gamma_1 = -1-i$ and $\gamma_2 = -i$ are Floquet exponents as well as eigenvalues of R which correspond to the Floquet multipliers $\lambda_1 = -2\pi$ and $\lambda_2 = 1$, respectively. That is, $e^{-2\pi-2\pi i} = e^{-2\pi}$ and $e^{-2\pi i} = 1$.

8.3 Time Scale Example

Finally, we consider the time scale example with $\mathbb{T} = \mathbb{P}_{1,1}$. The original matrix R' was found to be

$$R' = \begin{bmatrix} -3 & C \\ 0 & -3 \end{bmatrix},$$

where $C = -e^3 \int_0^2 e_{-3+\sin(2\pi\tau)}(2,\sigma(s))e_{-3}(s,0)\Delta s$. Again define $R := R' \ominus \mathring{\imath}\frac{2\pi k}{2} I = R' \ominus \mathring{\imath}\pi I$, with $k = 1$ and $p = 2$, as in Theorem 7.3. Then

$$R = \begin{bmatrix} -3 & C \\ 0 & -3 \end{bmatrix} \ominus \begin{bmatrix} \mathring{\imath}\pi & 0 \\ 0 & \mathring{\imath}\pi \end{bmatrix} = \begin{bmatrix} -3 \ominus \mathring{\imath}\pi & C \\ 0 & -3 \ominus \mathring{\imath}\pi \end{bmatrix},$$

and thus,

$$e_R(t,0) = \begin{bmatrix} e_{-3\ominus\mathring{\imath}\pi}(t,0) & C\int_0^t e_{-3\ominus\mathring{\imath}\pi}(t,\sigma(s))e_{-3\ominus\mathring{\imath}\pi}(s,0)\Delta s \\ 0 & e_{-3\ominus\mathring{\imath}\pi}(t,0) \end{bmatrix}.$$

Since $\int_0^2 e_{-3\ominus \mathring{\imath}\pi}(2,\sigma(s))e_{-3\ominus \mathring{\imath}\pi}(s,0)\Delta s = \int_0^2 e_{-3}(2,\sigma(s))e_{-3}(s,0)\Delta s = -e^{-3}$, this satisfies

$$\begin{aligned}
e_R(2,0) &= \begin{bmatrix} e_{-3\ominus \mathring{\imath}\pi}(2,0) & C\int_0^2 e_{-3\ominus \mathring{\imath}\pi}(2,\sigma(s))e_{-3\ominus \mathring{\imath}\pi}(s,0)\Delta s \\ 0 & e_{-3\ominus \mathring{\imath}\pi}(2,0) \end{bmatrix} \\
&= \begin{bmatrix} -2e^{-3} & C\int_0^2 e_{-3}(2,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & -2e^{-3} \end{bmatrix} \\
&= \begin{bmatrix} -2e^{-3} & \int_0^2 e_{-3+\sin(2\pi\tau)}(2,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & -2e^{-3} \end{bmatrix} \\
&= \Phi_A(2,0).
\end{aligned}$$

Next, recall that

$$L'(t) = \frac{1}{(e_{-3}(t,0))^2}\begin{bmatrix} e_{-3+\sin(2\pi t)}(t,0) & \int_0^t e_{-3+\sin(2\pi\tau)}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot$$
$$\begin{bmatrix} e_{-3}(t,0) & -C\int_0^t e_{-3}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix}.$$

Using the original Lyapunov transformation matrix $L'(t)$ from the time scale example, define

$$L(t) := L'(t)e_{\frac{\mathring{\imath}2\pi k}{2}I}(t,0) = L'(t)e_{\mathring{\imath}\pi I}(t,0).$$

Hence

$$L(t) = \frac{1}{(e_{-3}(t,0))^2}\begin{bmatrix} e_{-3+\sin(2\pi t)}(t,0) & \int_0^t e_{-3+\sin(2\pi\tau)}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot$$
$$\begin{bmatrix} e_{-3}(t,0) & -C\int_0^t e_{-3}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot \begin{bmatrix} e_{\overset{\circ}{i\pi}}(t,0) & 0 \\ 0 & e_{\overset{\circ}{i\pi}}(t,0) \end{bmatrix}$$
$$= \frac{1}{(e_{-3}(t,0))^2}\begin{bmatrix} e_{-3+\sin(2\pi t)}(t,0) & \int_0^t e_{-3+\sin(2\pi\tau)}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot$$
$$\begin{bmatrix} e_{-3\oplus\overset{\circ}{i\pi}}(t,0) & -C\int_0^t e_{-3}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3\oplus\overset{\circ}{i\pi}}(t,0) \end{bmatrix}.$$

Thus,

$$L'(t)e_{R'}(t,0) = L'(t)e_{R\oplus\overset{\circ}{i\pi}I}(t,0) = L'(t)e_{\overset{\circ}{i\pi}I}(t,0)e_R(t,0) = L(t)e_R(t,0),$$

and so we have

$$L'(t)e_{R'}(t,0) = \frac{1}{(e_{-3}(t,0))^2}\begin{bmatrix} e_{-3+\sin(2\pi t)}(t,0) & \int_0^t e_{-3+\sin(2\pi\tau)}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot$$
$$\begin{bmatrix} e_{-3}(t,0) & -C\int_0^t e_{-3}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot$$
$$\begin{bmatrix} e_{-3}(t,0) & C\int_0^t e_{-3}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix}$$

$$= \frac{1}{(e_{-3}(t,0))^2} \begin{bmatrix} e_{-3+\sin(2\pi t)}(t,0) & \int_0^t e_{-3+\sin(2\pi\tau)}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot$$

$$\begin{bmatrix} e_{-3}(t,0) & -C\int_0^t e_{-3}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot$$

$$\begin{bmatrix} e_{\mathring{i}\pi}(t,0) & 0 \\ 0 & e_{\mathring{i}\pi}(t,0) \end{bmatrix} \begin{bmatrix} e_{-3\ominus\mathring{i}\pi}(t,0) & C\int_0^t e_{-3}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3\ominus\mathring{i}\pi}(t,0) \end{bmatrix}$$

$$= \frac{1}{(e_{-3}(t,0))^2} \begin{bmatrix} e_{-3+\sin(2\pi t)}(t,0) & \int_0^t e_{-3+\sin(2\pi\tau)}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3}(t,0) \end{bmatrix} \cdot$$

$$\begin{bmatrix} e_{-3\oplus\mathring{i}\pi}(t,0) & -C\int_0^t e_{-3}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3\oplus\mathring{i}\pi}(t,0) \end{bmatrix} \cdot$$

$$\begin{bmatrix} e_{-3\ominus\mathring{i}\pi}(t,0) & C\int_0^t e_{-3}(t,\sigma(s))e_{-3}(s,0)\Delta s \\ 0 & e_{-3\ominus\mathring{i}\pi}(t,0) \end{bmatrix}$$

$$= L(t)e_R(t,0).$$

Therefore, $\Phi_A(t,0) = L(t)e_R(t,0)$ is another Floquet decomposition of the transition matrix and $\gamma = -3 \ominus \mathring{i}\pi$ is another Floquet exponent as well as an eigenvalue of R which corresponds to the Floquet multiplier $\lambda = -2e^{-3}$; that is $e_{-3\ominus\mathring{i}\pi}(2,0) = e_{-3}(2,0) = -2e^{-3}$.

CHAPTER NINE

Conclusions and Future Directions

This dissertation has presented a very general background on the rapidly growing area of mathematics known as dynamic equations on time scales. In particular, the focus has been on first order linear dynamic systems and the analysis of the system's stability characteristics via a generalized version of Lyapunov's direct (second) method. Stability properties of systems with periodic coefficient matrices on periodic time scales were also analyzed by a unified Floquet theory.

There are many possibilities for applications of time scales theory. The papers by Gravagne, Davis, DaCunha, and Marks [20, 21] demonstrate the use of time scales in high gain adaptive control and bandwidth reduction. The theory offers a cleaner way to unify the disparate cases of discrete and continuous sampling. The stability theory introduced in this dissertation has aided in the development of these applications.

The Floquet theory that has been unified also has many possible avenues for investigation and analysis. Applying the unified Floquet theory to switched linear systems as in [18], as well as almost periodic systems on time scales such as [29], are specific areas of interest.

More investigation and development needs to be done on the time scale exponential function, as well as the time scale matrix exponential function and transition matrix. The Putzer Algorithm [1, 6] does give a way to calculate the matrix exponential for an individual matrix, however, there does not exist a closed form solution for $e_A(t, t_0)$ nor $\Phi_A(t, t_0)$. Perhaps some generalization of the Peano-Baker series [42] could be of use in finding these closed forms. In addition to the matrix exponential and transition matrix, to the author's knowledge, there is virtually nothing in the literature that gives any insight to a generalized version of a time scales logarithm.

In the coming fall, the author will investigate applications of time scales to unmanned autonomous vehicles (UAVs) and unmanned ground vehicles (UGVs) at the United States Military Academy in West Point, New York and the Army Research Laboratory on the Aberdeen Proving Grounds in Aberdeen, Maryland.

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