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ADAPTIVE CONTROL IN THE NABLA SETTING

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Abstract. In this paper, we develop notions of Lyapunov stability for the nabla time scale exponential function. We begin by reviewing some of the necessary prerequisite definitions and theorems for nabla differential equations. We then proceed to discuss the stability of the ordinary dynamic equation (ODE) that defines the nabla exponential function. We conclude with a state feedback result showing that the arbitrary linear ODE can be stabilized by using the controllability Gramian.

1. Introduction

The theory of time scales originated in Stefan Hilger's dissertation [12] that evolved into his seminal paper on the subject [11]. Originally intended to unify continuous and discrete analysis, the theory has gone well beyond this aspect into extension of familiar properties of dynamic equations on arbitrary domains. Recently, time scales analysis has received a considerable amount of attention in the context of engineering applications, particularly in systems theory and control (see [8, 9, 10]). These results on stability and control have dealt almost solely with the delta (forward) derivative.

Here, we wish to establish analogous results for the nabla (backward) derivative. The utility of such an analysis becomes evident when one considers that the time scales analysis could also have important implications for numerical analysts, who often use backward differences rather than forward differences to handle their computations.

With this in mind, we begin with a review of the appropriate time scale definitions and theorems in the nabla setting. The interested reader is urged to examine the works of Bohner and Peterson in [1, 2].

2. Background

We first review several definitions and theorems about the nabla derivative.

Definition 2.1. Let $\mathbb{T}$ be a nonempty closed subset of the reals, called a **time scale**. For each $\mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$, the following are defined:

(i) The **backward jump operator** $\rho : \mathbb{T} \to \mathbb{T}$ is given by
$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$
If $\rho(t) = t$, then t is **left dense**: otherwise t is **left scattered**.

(ii) The **backward graininess** $\nu : \mathbb{T} \to \mathbb{R}$ is defined by
$$\nu(t) = t - \rho(t).$$

(iii) The **nabla derivative** $f^\nabla(t)$ of $f : \mathbb{T} \to \mathbb{R}$ is the quantity (provided it exists)
$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$
In this definition, if $\nu(t) = 0$ (i.e. if t is left dense) then this quantity is interpreted in the limit sense as $\nu \to 0$.

(iv) $f : \mathbb{T} \to \mathbb{R}$ is said to be **left dense continuous** (abbreviated ld-continuous) if $f(t)$ exists for all $t \in \mathbb{T}$ and f is continuous from the left at left dense points of $\mathbb{T}$.

(v) For $f(t)$ a ld-continuous function, suppose there exists a function $F(t)$ with $F^\nabla(t) = f(t)$. Then the **nabla integral** of $f(t)$ is given by
$$\int f(t)\nabla t = F(t) + c.$$

Key words and phrases. nabla derivative, Hilger circle, Lyapunov stability, controllability Gramian.

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Theorem 2.1. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_\kappa$. Then:*

(i) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with*

$$(f+g)^\nabla(t) = f^\nabla(t) + g^\nabla(t).$$

(ii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t , and we get the product rules*

$$(fg)^\nabla(t) = f^\nabla(t)g(t) + f(\rho(t))g^\nabla(t) = f(t)g^\nabla(t) + f^\nabla(t)g(\rho(t)).$$

(iii) *If $g(t)g(\rho(t)) \neq 0$, then f/g is nabla differentiable at t, and we get the quotient rule*

$$\left(\frac{f}{g}\right)^\nabla(t) = \frac{f^\nabla(t)g(t) - f(t)g^\nabla(t)}{g(t)g(\rho(t))}.$$

(iv) *If f and $f^\nabla(t)$ are continuous, then*

$$\left(\int_a^t f(t,s)\nabla s\right)^\nabla = f(\rho(t),t) + \int_a^t f^{\nabla_t}(t,s)\nabla s.$$

Definition 2.2. The function $p : \mathbb{T} \to \mathbb{R}$ is ν-regressive if

$$1 - \nu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}_\kappa.$$

The ν-regressive group $(\mathcal{R}_\nu, \oplus_\nu, \ominus_\nu)$ is the set

$$\mathcal{R}_\nu = \{p : \mathbb{T} \to \mathbb{R} : p \text{ is ld-continuous and } \nu\text{-regressive}\},$$

together with the operations

$$p \oplus_\nu q = p + q - \nu pq$$

and

$$\ominus_\nu p = -\frac{p}{1-\nu p}.$$

p is positively ν-regressive if

$$1 - \nu p > 0.$$

Definition 2.3. For $p \in \mathcal{R}_\nu$, the unique solution to the equation

$$y^\nabla(t) = p(t)y(t), \quad y(t_0) = 1,$$

is called the *nabla time scale exponential function* and is denoted by $y(t) = \hat{e}_p(t, t_0)$. The nabla exponential function has closed form

$$\hat{e}_p(t,t_0) = \exp\left(\int_{t_0}^t -\frac{\text{Log}(1-\nu(\tau)p(\tau))}{\nu(\tau)}\nabla\tau\right).$$

Theorem 2.2 (Properties of the Nabla Exponential)**.** *Let $p, q \in \mathcal{R}_\nu$ and $s, t, r \in \mathbb{T}$. Then*

(i) $\hat{e}_0(t,s) \equiv 1$ *and* $\hat{e}_p(t,t) \equiv 1$*;*
(ii) $\hat{e}_p(\rho(t),s) = (1-\nu(t)p(t))\hat{e}_p(t,s)$*;*
(iii) $\frac{1}{\hat{e}_p(t,s)} = \hat{e}_{\ominus_\nu p}(t,s)$*;*
(iv) $\hat{e}_p(t,s) = \frac{1}{\hat{e}_p(s,t)} = \hat{e}_{\ominus_\nu p}(s,t)$*;*
(v) $\hat{e}_p(t,r)\hat{e}_p(r,s) = \hat{e}_p(t,s)$*;*
(vi) $\hat{e}_p(t,r)\hat{e}_q(t,r) = \hat{e}_{p\oplus_\nu q}(t,r)$*;*
(vii) $\frac{\hat{e}_p(t,s)}{\hat{e}_q(t,s)} = \hat{e}_{p\ominus_\nu q}(t,s)$*;*
(viii) $\left(\frac{1}{\hat{e}_p(t,s)}\right)^\nabla = -\frac{p(t)}{\hat{e}_p(\rho(t),s)}$*;*
(ix) *If p is positively ν-regressive, then $\hat{e}_p(t,t_0) > 0$.*

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3. Stability of the Nabla Exponential

A natural question is the following: For what $z \in \mathbb{C}$ does it follow that

$$\lim_{t\to\infty} \hat{e}_z(t,t_0) = 0?$$

If we examine the closed form of the nabla exponential, then a sufficient collection of such $z \in \mathbb{C}$ would be the set

$$\left\{ z \in \mathbb{C} : \left| z - \frac{1}{\nu(t)} \right| > \frac{1}{\nu(t)} \right\}.$$

(For the corresponding result in the delta case, see [3, 4, 7].) We will call the set

$$\mathbb{H}_\nu := \left\{ z \in \mathbb{C} : \left| z - \frac{1}{\nu(t)} \right| = \frac{1}{\nu(t)} \right\}$$

the ν-Hilger circle due to its importance in determining exponential stability.

We would like a geometric interpretation and connection of the set of exponential stability akin to the one known for the delta case (see [1]). To do this, we will need to define the ν-Hilger complex plane.

Definition 3.1 (ν-Hilger Complex Plane)**.** For $\nu > 0$ we define the *ν-Hilger complex numbers*, the *ν-Hilger real axis*, the *ν-Hilger alternating axis*, and the *ν-Hilger imaginary circle* as

$$\begin{aligned}
\mathbb{C}_\nu &:= \left\{ z \in \mathbb{C} : z \neq \frac{1}{\nu} \right\}, \\
\mathbb{R}_\nu &:= \left\{ z \in \mathbb{C}_\nu : z \in \mathbb{R} \text{ and } z < \frac{1}{\nu} \right\}, \\
\mathbb{A}_\nu &:= \left\{ z \in \mathbb{C}_\nu : z \in \mathbb{R} \text{ and } z > \frac{1}{\nu} \right\}, \\
\mathbb{I}_\nu &:= \left\{ z \in \mathbb{C}_\nu : \left| z - \frac{1}{\nu} \right| = \frac{1}{\nu} \right\} = \mathbb{H}_\nu,
\end{aligned}$$

respectively. For $h = 0$, let $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{I}_0 := i\mathbb{R}$, and $\mathbb{A}_0 := \emptyset$.

Definition 3.2 (The ν-Hilger Complex plane)**.** Let $\nu > 0$ and $z \in \mathbb{C}_\nu$. We define the ν-Hilger real part of z by

$$\mathrm{Re}_\nu(z) := \frac{1 - |1 - \nu z|}{\nu}$$

and the ν-Hilger imaginary part of z by

$$\mathrm{Im}_\nu(z) := -\frac{\mathrm{Arg}(1 - z\nu)}{\nu},$$

where $\mathrm{Arg}(z)$ denotes the principal argument of z (i.e., $-\pi < \mathrm{Arg}(z) \leq \pi$). For $-\frac{\pi}{\nu} \leq \omega < \frac{\pi}{\nu}$, we define the ν-Hilger purely imaginary number $\mathring{\imath}\omega$ by

$$\mathring{\imath}\omega = \frac{1 - e^{-i\omega\nu}}{\nu}.$$

Note that $\mathrm{Re}_\nu(z)$ and $\mathrm{Im}_\nu(z)$ satisfy

$$-\infty < \mathrm{Re}_\nu(z) < \frac{1}{\nu} \quad \text{and} \quad -\frac{\pi}{\nu} \leq \mathrm{Im}_\nu(z) < \frac{\pi}{\nu},$$

respectively. In particular, $\mathrm{Re}_\nu(z) \in \mathbb{R}_\nu$. Also, for $z \in \mathbb{C}_\nu$, we have that $\mathring{\imath}\mathrm{Im}_\nu(z) \in \mathbb{H}_\nu$. The ν-Hilger complex plane can be seen in Figure 1.

Theorem 3.1. *For $z \in \mathbb{C}_\nu$ we have*

$$z = Re_\nu(z) \oplus_\nu \mathring{\imath} Im_\nu(z).$$

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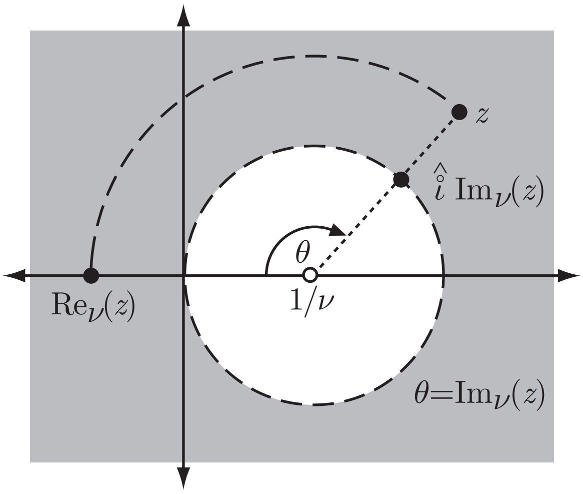

FIGURE 1. The ν-Hilger Complex Plane. Points interior to the ν-Hilger circle $\mathbb{H}_\nu$ have positive ν-Hilger real part, while points exterior to the circle have negative ν-Hilger real part. Points on the circle therefore have zero ν-Hilger real part. The shading indicates that points exterior to the largest ν-Hilger circle (i.e. the one corresponding to ν_*) lie in the stability region.

Proof. Let $z \in \mathbb{C}_\nu$. Then

$$\begin{aligned}
\mathrm{Re}_\nu(z) \oplus_\nu \hat{\imath}\mathrm{Im}_\nu(z) &= \frac{1-|1-z\nu|}{\nu} \oplus_\nu \hat{\imath}\left(-\frac{\mathrm{Arg}(1-z\nu)}{\nu}\right) \\
&= \frac{1-|1-z\nu|}{\nu} \oplus_\nu \frac{1-\exp(i\mathrm{Arg}(1-z\nu))}{\nu} \\
&= \frac{1-|1-z\nu|}{\nu} + \frac{1-\exp(i\mathrm{Arg}(1-z\nu))}{\nu} - \nu\frac{1-|1-z\nu|}{\nu}\frac{1-\exp(i\mathrm{Arg}(1-z\nu))}{\nu} \\
&= \frac{1}{\nu}\left\{1-|1-z\nu|\exp(i\mathrm{Arg}(1-z\nu))\right\} \\
&= \frac{1-(1-z\nu)}{\nu} = z.
\end{aligned}$$

□

Notice that as we stated before, the stability region is cast in terms of $\mathbb{H}_\nu$. Points in the stability region that we have chosen always have negative ν-Hilger real part. (Note that we often abuse the notation and say that points in the stability region lie in the ν-Hilger circle when actually they are exterior to the largest ν-Hilger circle corresponding to $\nu_{min} = \nu_*$.) We could extend our stability region by considering points for which the ν-Hilger real part is negative on average as Pötsche, Siegmund, and Wirth do for the delta case in [16], but for our purposes the Hilger circle will suffice for stability.

It is also worth noting that for points $z = \hat{\imath}\omega$ on the ν-Hilger circle, we have

$$\left|\hat{e}_{\hat{\imath}\omega}(t,t_0)\right| = 1.$$

Further, the ν-Hilger real axis is so named because for points $c < \frac{1}{\nu}$ on this axis, we have $\hat{e}_c(t,t_0) > 0$, while for points on the ν-Hilger alternating axis, we have that the nabla exponential is real valued and changes sign at every point. The nabla exponential is never zero for any regressive subscript. Finally, the positively regressive constants for the nabla exponential are simply the negative real axis.

As $\nu \to 0$, we see that the ν-Hilger circle tends to the open left-half plane as we would expect since for $\mathbb{T} = \mathbb{R}$ (where $\nu \equiv 0$), the time scale exponential function is the continuous exponential (i.e. $\hat{e}_z(t,0) = e^{zt}$). As $\nu \to 1$, we see that the stability region tends to the exterior of a circle of unit radius centered at $z = 1$.

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This should also make sense because for $\mathbb{T} = \mathbb{Z}$, we have $\hat{e}_z(t,0) = (1-z)^{-t}$. However, notice in general that the ν-Hilger circle is *dynamic*, varying as ν varies over $\mathbb{T}$. Thus, in some sense, exponential stability becomes a "moving target".

4. Gronwall's Inequality For the Nabla Integral

We shall need Gronwall's inequality for later results, so we state and prove it here. (Actually, the proofs that follow mirror their delta counterparts given in [1], but we give them here for the sake of completeness.)

Theorem 4.1. *Let $y, f \in C_{ld}$ and $p \in \mathcal{R}_\nu^+$. Then*

$$y^\nabla(t) \le p(t)y(t) + f(t) \quad \text{for all} \quad t \in \mathbb{T}$$

implies

$$y(t) \le y(t_0)\hat{e}_p(t,t_0) + \int_{t_0}^t \hat{e}_p(t,\rho(\tau))f(\tau)\nabla\tau \quad \text{for all} \quad t \in \mathbb{T}.$$

Proof. We use the product rule and Theorem 2.2 (ii) to calculate

$$\begin{aligned}
\left[y\hat{e}_{\ominus_\nu p}(\cdot,t_0)\right]^\nabla(t) &= y^\nabla(t)\hat{e}_{\ominus_\nu p}(\rho(t),t_0) + y(t)(\ominus_\nu p)(t)\hat{e}_{\ominus_\nu p}(t,t_0) \\
&= y^\nabla(t)\hat{e}_{\ominus_\nu p}(\rho(t),t_0) + y(t)\frac{\ominus_\nu p)(t)}{1-\nu(t)(\ominus_\nu p)(t)}\hat{e}_{\ominus_\nu p}(\rho(t),t_0) \\
&= \left[y^\nabla(t) - (\ominus_\nu(\ominus_\nu p))(t)y(t)\right]\hat{e}_{\ominus_\nu p}(\rho(t),t_0) \\
&= \left[y^\nabla(t) - p(t)y(t)\right]\hat{e}_{\ominus_\nu p}(\rho(t),t_0).
\end{aligned}$$

Since $p \in \mathcal{R}_\nu^+$, $\ominus_\nu p \in \mathcal{R}_\nu^+$ since the positively ν-regressive functions are a subgroup of the ν-regressive functions. Thus, $\hat{e}_{\ominus_\nu p} > 0$ by Theorem 2.2 (ix). Now

$$\begin{aligned}
y(t)\hat{e}_{\ominus_\nu p}(t,t_0) - y(t_0) &= \int_{t_0}^t \left[y^\nabla(\tau) - p(\tau)y(\tau)\right]\hat{e}_{\ominus_\nu p}(\rho(\tau),t_0)\nabla\tau \\
&\le \int_t^{t_0} f(\tau)\hat{e}_{\ominus_\nu p}(\rho(\tau),t_0)\nabla\tau \\
&= \int_{t_0}^t \hat{e}_p(t_0,\rho(\tau))f(\tau)\nabla\tau,
\end{aligned}$$

and hence the assertion follows by applying Theorem 2.2. □

Theorem 4.2 (Bernoulli's Inequality.)**.** *Let $\alpha \in \mathbb{R}$ with $\alpha \in \mathcal{R}_\nu^+$. Then*

$$\hat{e}_\alpha(t,s) \ge 1 + \alpha(t-s) \quad \text{for all} \quad t \ge s.$$

Proof. Since $\alpha \in \mathcal{R}_\nu^+$, we have $\hat{e}_\alpha(t,s) > 0$ for all $t, s \in \mathbb{T}$. Suppose $t, s \in \mathbb{T}$ with $t \ge s$. Let $y(t) = \alpha(t-s)$. Then

$$\alpha y(t) + \alpha = \alpha^2(t-s) + \alpha \ge \alpha = y^\nabla(t).$$

Since $y(s) = 0$, we have by Theorem 4.1 (with $p(t) = f(t) \equiv \alpha$)

$$y(t) \le \int_s^t \hat{e}_\alpha(t,\rho(\tau))\alpha\nabla\tau = \hat{e}_\alpha(t,s) - 1.$$

Hence, $\hat{e}_\alpha(t,s) \ge 1 + y(t) = 1 + \alpha(t-s)$ follows. □

Theorem 4.3 (Gronwall's Inequality.)**.** *Let $y, f \in C_{ld}$ and $p \in \mathcal{R}_\nu^+$, $p \ge 0$. Then*

$$y(t) \le f(t) + \int_{t_0}^t y(\tau)p(\tau)\nabla\tau \quad \text{for all} \quad t \in \mathbb{T}$$

implies

$$y(t) \le f(t) + \int_{t_0}^t \hat{e}_p(t,\rho(\tau))f(\tau)p(\tau)\nabla\tau \quad \text{for all} \quad t \in \mathbb{T}.$$

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Proof. Define

$$z(t)=\int_{t_0}^{t} y(\tau)p(\tau)\nabla\tau.$$

Then $z(t_0)=0$ and

$$z^{\nabla}(t)=y(t)p(t)\le [f(t)+z(t)]\,p(t)=p(t)z(t)+p(t)f(t).$$

By Theorem 4.1,

$$z(t)\le\int_{t_0}^{t}\hat{e}_p(t,\rho(\tau))f(\tau)p(\tau)\nabla\tau.$$

and hence the claim follows because of $y(t)\le f(t)+z(t)$. □

Corollary 4.1. *Let $y\in C_{ld}$ and $p\in\mathcal{R}_{\nu}^{+}$ with $p\ge 0$. Then*

$$y(t)\le\int_{t_0}^{t} y(\tau)p(\tau)\nabla\tau \quad \textit{for all} \quad t\in\mathbb{T}$$

implies

$$y(t)\le 0 \quad \textit{for all} \quad t\in\mathbb{T}.$$

Proof. This is Theorem 4.3 with $f(t)\equiv 0$. □

Corollary 4.2. *Let $y\in C_{ld}$, $p\in\mathcal{R}_{\nu}^{+}, p\ge 0$, and $\alpha\in\mathbb{R}$. Then*

$$y(t)\le\alpha+\int_{t_0}^{t} y(\tau)p(\tau)\nabla\tau \quad \textit{for all} \quad t\in\mathbb{T}$$

implies

$$y(t)\le\alpha\hat{e}_p(t,t_0) \quad \textit{for all} \quad t\in\mathbb{T}.$$

Proof. In Theorem 4.3, let $f(t)\equiv\alpha$. Then by Theorem 4.3,

$$\begin{aligned} y(t) &\le \alpha+\int_{t_0}^{t}\hat{e}_p(t,\rho(\tau))\alpha p(\tau)\nabla\tau \\ &= \alpha\left[1+\int_{t_0}^{t}p(\tau)\hat{e}_p(t,\rho(\tau))\nabla\tau\right] \\ &= \alpha[1+\hat{e}_p(t,t_0)-\hat{e}_p(t,t)] \\ &= \alpha\hat{e}_p(t,t_0). \end{aligned}$$

Thus, the claim follows. □

Corollary 4.3. *Let $y\in C_{ld}$ and $\alpha,\beta,\gamma\in\mathbb{R}$ with $\gamma>0$. Then*

$$y(t)\le\alpha+\beta(t-t_0)+\gamma\int_{t_0}^{t} y(\tau)\nabla\tau \quad \textit{for all} \quad t\in\mathbb{T}$$

implies

$$y(t)\le\left(\alpha+\frac{\beta}{\gamma}\right)\hat{e}_{\gamma}(t,t_0)-\frac{\beta}{\gamma} \quad \textit{for all} \quad t\in\mathbb{T}.$$

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Proof. In Theorem 4.3, let $f(t) = \alpha + \beta(t - t_0)$ and $p(t) \equiv \gamma$. Note that for $w(\tau) = \hat{e}_\gamma(t, \tau)$ we have $w^\nabla(\tau) = -\gamma\hat{e}_\gamma(t, \rho(\tau))$. By Theorem 4.3,

$$\begin{aligned} y(t) &\leq f(t) + \int_{t_0}^{t} \hat{e}_\gamma(t, \rho(\tau))\gamma f(\tau)\nabla\tau \\ &= f(t)w(t) - \int_{t_0}^{t} w^\nabla(\tau)f(\tau)\nabla\tau \\ &= f(t_0)w(t_0) + \int_{t_0}^{t} w(\rho(\tau))f^\nabla(\tau)\nabla\tau \\ &= \alpha\hat{e}_\gamma(t, t_0) + \int_{t_0}^{t} \hat{e}_\gamma(t, \rho(\tau))\beta\nabla\tau \\ &= \alpha\hat{e}_\gamma(t, t_0) + \frac{\beta}{\gamma}\int_{t_0}^{t} \gamma\hat{e}_\gamma(t, \rho(\tau))\nabla\tau \\ &= \alpha\hat{e}_\gamma(t, t_0) + \frac{\beta}{\gamma}(\hat{e}_\gamma(t, t_0) - 1). \end{aligned}$$

Hence, the claim follows. □

5. The Systems Case

We now wish to turn our attention to the systems case. As with the scalar case, we begin by reviewing some of the pertinent definitions and results that we will need later.

Definition 5.1. Let $\mathbb{A}$ be an $n \times n$-matrix-valued function on $\mathbb{T}$. A is ld-continuous if every entry of A is ld-continuous. The class of all ld-continuous matrices is denoted by

$$C_{\mathrm{ld}} = C_{\mathrm{ld}}(\mathbb{T}) = C_{\mathrm{ld}}(\mathbb{T}, \mathbb{R}^{m\times n}).$$

A is nabla differentiable on $\mathbb{T}$ if every entry of A is nabla differentiable on $\mathbb{T}$, in which case

$$A^\nabla(t) = (a_{ij}^\nabla(t))_{1\leq i\leq n, 1\leq j\leq n}.$$

We say A is ν-regressive if

$$I - \nu(t)A(t) \quad \text{is invertible for all} \quad t \in \mathbb{T}_\kappa,$$

and the class of all such ν-regressive and ld-continuous matrix functions is denoted by

$$\mathcal{R}_\nu = \mathcal{R}_\nu(\mathbb{T}) = \mathcal{R}_\nu(\mathbb{T}, \mathbb{R}^{n\times n}).$$

The system

$$x^\nabla(t) = A(t)x(t), \quad x(t_0) = x_0,$$

is called ν-regressive if A is ν-regressive.

Theorem 5.1. *Suppose A and B are nabla differentiable $n \times n$-matrix-valued functions. Then*

(i) $(A + B)^\nabla(t) = A^\nabla(t) + B^\nabla(t)$;
(ii) $(\alpha A)^\nabla(t) = \alpha A^\nabla(t)$ *if α is constant;*
(iii) $(AB)^\nabla(t) = A^\nabla(t)B(\rho(t)) + A(t)B^\nabla(t) = A(\rho(t))B^\nabla(t) + A^\nabla(t)B(t)$;
(iv) $(A^{-1})^\nabla = -(A(\rho(t)))^{-1}A^\nabla(t)A^{-1}(t) = -A^{-1}(t)A^\nabla(t)(A(\rho(t)))^{-1}$ *if $A(t)A(\rho(t))$ is invertible.*
(v) $(AB^{-1})^\nabla(t) = (A^\nabla(t) - A(t)B^{-1}(t)B^\nabla(t))(B(\rho(t)))^{-1} = (A^\nabla(t) - A(\rho(t))B^{-1}(\rho(t))B^\nabla(t))B^{-1}(t)$ *if $B(t)B(\rho(t))$ is invertible.*

Definition 5.2. The ν-regressive group $(\mathcal{R}_\nu(\mathbb{T}, \mathbb{R}^{n\times n}), \oplus_\nu, \ominus_\nu)$ is the set

$$\mathcal{R}_\nu(\mathbb{T}, \mathbb{R}^{n\times n}) = \{A \in R^{n\times n} : A \text{ is regressive and ld-continuous}\}$$

together with the operation $\oplus_\nu$ defined by

$$A \oplus_\nu B := A + B - \nu AB,$$

and inverse operation $\ominus_\nu$ given by

$$\ominus_\nu A = -A(I - \nu A)^{-1}.$$

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We are interested in solutions to nabla dynamic equations. We shall denote the solution of

$$Y^{\nabla}(t) = A(t)Y(t), \quad Y(t_0) = I$$

as $Y(t) = \hat{\phi}_A(t, t_0)$.

Theorem 5.2 (Variation of Parameters.). *Let $A \in \mathcal{R}_\nu(\mathbb{T}, \mathbb{R}^{n\times n})$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is ld-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^{\nabla}(t) = A(t)y(t) + f(t), \quad y(t_0) = y_0,$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$ given by

$$y(t) = \hat{\phi}_A(t, t_0)y_0 + \int_{t_0}^{t} \hat{\phi}_A(t, \rho(\tau))f(\tau)\nabla\tau.$$

6. Exponential Stability and Lyapunov Criteria

We seek conditions that guarantee that solutions of

$$x^{\nabla}(t) = A(t)x(t), \quad x(t_0) = x_0$$

tend to zero as $t \to \infty$. That is, we wish to establish a notion of *asymptotic stability* for this equation. For our purposes, *uniform exponential stability* will suffice, so we define this notion here. For the reader interested in the analogous results for the delta case, see [3, 4].

Definition 6.1. The time varying ν-regressive linear nabla dynamic equation

$$x^{\nabla}(t) = A(t)x(t), \quad x(t_0) = x_0$$

is said to be *uniformly exponentially stable* if there exist constants $\gamma, \lambda > 0$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$||x(t)|| \leq ||x(t_0)||\gamma\hat{e}_{-\lambda}(t, t_0), \quad t \geq t_0.$$

We make the blanket assumption that $\mathbb{T}$ is unbounded above. We associate with the state equation the scalar function

$$||x(t)||^2 = x^T(t)x(t)$$

that acts as the system's associated energy function. We want conditions on our system that guarantee that $||x(t)||^2 \to 0$ as $t \to \infty$. We begin by noting that the energy function has time nabla derivative

$$\begin{aligned}
(||x(t)||^2)^{\nabla_t} &= (x^T(t)x(t))^{\nabla_t} \\
&= x^{T^{\nabla}}(t)x(t) + x^{T^{\rho}}(t)x^{\nabla}(t) \\
&= x^T(t)A^T(t)x(t) + x^T(t)(I - \nu(t)A^T(t))A(t)x(t) \\
&= x^T(t)[A^T(t) + A(t) - \nu(t)A^T(t)A(t)]x(t).
\end{aligned}$$

Thus, if the quadratic form we obtain from the derivative is negative definite, then we will have $||x(t)||^2 \to 0$ as $t \to \infty$, as desired. From this discussion, we see that if we can establish the existence of a symmetric matrix $Q(t) \in C^1_{\mathrm{ld}}(\mathbb{T}, \mathbb{R}^{n\times n})$ such that

$$\begin{aligned}
[x^T(t)Q(t)x(t)]^{\nabla_t} &= x^{T^{\nabla}}(t)Q(t)x(t) + x^{T^{\rho}}(t)(Q^{\nabla}(t)x^{\rho}(t) + Q(t)x^{\nabla}(t)) \\
&= x^T(t)[A^T(t)Q(t) + (I - \nu(t)A^T(t))Q^{\nabla}(t)(I - \nu(t)A(t)) \\
&\quad +(I - \nu(t)A^T(t))Q(t)A(t)]x(t)
\end{aligned}$$

is negative definite, then we get asymptotic decay. We shall need other versions of the derivative of the quadratic functional given above, so we present them here. Note that

$$\begin{aligned}
[x^T(t)Q(t)x(t)]^{\nabla} &= (x^T(t)Q(t))^{\nabla}x^{\rho}(t) + x^T(t)Q(t)x^{\nabla}(t) \\
&= x^T(t)[A^T(t)Q^{\rho}(t)(I - \nu(t)A(t)) + Q^{\nabla}(t)(I - \nu(t)A(t)) + Q(t)A(t)]x(t),
\end{aligned}$$

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and also

$$\begin{aligned}
[x^T(t)Q(t)x(t)]^\nabla &= x^{T^\nabla}(t)Q(t)x(t) + x^{T^\rho}(t)(Q^\nabla(t)x^\rho(t) + Q(t)x^\nabla(t)) \\
&= \frac{1}{\nu(t)}(x^T(t) - x^{T^\rho}(t))Q(t)x(t) + \frac{1}{\nu(t)}x^{T^\rho}(t)(Q(t) - Q^\rho(t))x^\rho(t) \\
&\quad + \frac{1}{\nu(t)}x^{T^\rho}(t)Q(t)(x(t) - x^\rho(t)) \\
&= x^T(t)\left[\frac{Q(t) - (I - \nu(t)A^T(t))Q^\rho(t)(I - \nu(t)A(t))}{\nu(t)}\right]x(t).
\end{aligned}$$

Theorem 6.1 (Lyapunov Stability Criterion I)**.** *The time varying regressive nabla linear dynamic system*

$$x^\nabla(t) = A(t)x(t), \quad x(t_0) = x_0$$

is uniformly exponentially stable if there exists a symmetric matrix $Q(t) \in C^1_{ld}(\mathbb{T}, \mathbb{R}^{n\times n})$ *such that for all* $t \in \mathbb{T}$

(i) $\eta I \leq Q(t) \leq \kappa I$,
(ii) $A^T(t)Q(t) + (I - \nu(t)A^T(t))Q^\nabla(t)(I - \nu(t)A(t)) + (I - \nu(t)A^T(t))Q(t)A(t) \leq -\gamma I$,

where $\eta, \kappa, \gamma > 0$.

Proof. For any initial condition t_0 and $x(t_0) = x_0$ with corresponding solution $x(t)$ of the state equation, we see that for all $t \geq t_0$, (ii) gives

$$[x^T(t)Q(t)x(t)]^\nabla \leq -\gamma||x(t)||^2.$$

Also, for all $t \geq t_0$, (i) implies

$$x^T(t)Q(t)x(t) \leq \kappa||x(t)||^2.$$

Thus,

$$[x^T(t)Q(t)x(t)]^\nabla \leq -\frac{\gamma}{\kappa}x^T(t)Q(t)x(t)$$

for all $t \geq t_0$. Since $-\frac{\gamma}{\kappa} \in \mathcal{R}^+_\nu$, we can employ Theorem 4.1 to obtain

$$x^T(t)Q(t)x(t) \leq x^T(t_0)Q(t_0)x(t_0)\hat{e}_{-\gamma/\kappa}(t, t_0), \quad t \geq t_0. \tag{6.1}$$

By (i), $\eta I \leq Q(t)$ so that $\eta||x(t)||^2 \leq x^T(t)Q(t)x(t)$, and thus an application of (6.1) yields

$$||x(t)||^2 \leq \frac{1}{\eta}x^T(t)Q(t)x(t) \leq \frac{1}{\eta}x^T(t_0)Q(t_0)x(t_0)\hat{e}_{-\gamma/\kappa}(t, t_0), \quad t \geq t_0.$$

Now, $x(t_0)Q(t_0)x(t_0) \leq \kappa||x(t_0)||^2$ implies

$$||x(t)||^2 \leq \frac{\kappa}{\eta}||x(t_0)||^2\hat{e}_{-\gamma/\kappa}(t, t_0),$$

which yields

$$||x(t)|| \leq ||x(t_0)||\sqrt{\frac{\kappa}{\eta}\hat{e}_{-\gamma/\kappa}(t, t_0)}, \quad t \geq t_0.$$

Since this is true for arbitrary t_0 and $x(t_0)$, uniform exponential stability is established. □

If we use the other two representations of the derivative given above, then we see the proofs of the following two theorems are the same as the same as the previous one.

Theorem 6.2 (Lyapunov Stability Criterion II)**.** *The time varying regressive nabla linear dynamic system*

$$x^\nabla(t) = A(t)x(t), \quad x(t_0) = x_0$$

is uniformly exponentially stable if there exists a symmetric matrix $Q(t) \in C^1_{ld}(\mathbb{T}, \mathbb{R}^{n\times n})$ *such that for all* $t \in \mathbb{T}$

(i) $\eta I \leq Q(t) \leq \kappa I$,
(ii) $A^T(t)Q^\rho(t)(I - \nu(t)A(t)) + Q^\nabla(t)(I - \nu(t)A(t)) + Q(t)A(t) \leq -\gamma I$,

where $\eta, \kappa, \gamma > 0$.

Theorem 6.3 (Lyapunov Stability Criterion III). *The time varying regressive nabla linear dynamic system*

$$x^{\nabla}(t) = A(t)x(t), \quad x(t_0) = x_0$$

is uniformly exponentially stable if there exists a symmetric matrix $Q(t) \in C^1_{ld}(\mathbb{T}, \mathbb{R}^{n \times n})$ *such that for all* $t \in \mathbb{T}$

(i) $\eta I \leq Q(t) \leq \kappa I$,
(ii) $\left(Q(t) - (I - \nu(t)A^T(t))Q^{\rho}(t)(I - \nu(t)A(t))\right)/\nu(t) \leq -\gamma I$,

where $\eta, \kappa, \gamma > 0$.

7. Control and State Feedback

We desire an analogue of the feedback result obtained in [13] for the nabla dynamic equation. To do that, we first need to discuss controllability. The reader can see [6, 13] for the control results concerning the delta derivative, and [5, 14, 15, 17, 18] for the control and feedback theorems stated and proved for the special cases $\mathbb{T} = R$ and $\mathbb{T} = \mathbb{Z}$.

Definition 7.1. The ν-regressive linear nabla dynamic state equation

$$\begin{aligned} x^{\nabla}(t) &= A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0, \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{7.1}$$

is called *controllable* on $[t_0, t_f]_{\mathbb{T}}$ if given any initial state x_0 there exists a ld-continuous input signal $u(t)$ such that the corresponding solution of the system satisfies $x(t_f) = x_f$.

Theorem 7.1. *The ν-regressive nabla linear state equation (7.1) is controllable on* $[t_0, t_f]_{\mathbb{T}}$ *if and only if the* $n \times n$ controllability Gramian matrix

$$\hat{\mathcal{G}}_C(t_0, t_f) = \int_{t_0}^{t_f} \hat{\phi}_A(t_0, \rho(s))B(s)B^T(s)\hat{\phi}_A^T(t_0, \rho(s))\nabla s$$

is invertible.

Proof. Suppose $\hat{\mathcal{G}}_C(t_0, t_f)$ is invertible. Then, given x_0 and x_f, we can choose the input signal $u(t)$ as

$$u(t) = -B^T(t)\hat{\phi}_A(t_0, \rho(t))\hat{\mathcal{G}}_C^{-1}(t_0, t_f)(x_0 - \hat{\phi}_A(t_0, t_f)x_f), \qquad t \in (t_0, t_f],$$

and extend $u(t)$ continuously for all other values of t. The corresponding solution of the system at $t = t_f$ can be written as

$$\begin{aligned} x(t_f) &= \hat{\phi}_A(t_f, t_0)x_0 + \int_{t_0}^{t_f} \hat{\phi}_A(t_f, \rho(s))B(s)u(s)\nabla s \\ &= \hat{\phi}_A(t_f, t_0)x_0 - \int_{t_0}^{t_f} \hat{\phi}_A(t_f, \rho(s))B(s)B^T(s)\hat{\phi}_A^T(t_f, \rho(s))\hat{\mathcal{G}}_C^{-1}(t_0, t_f)(x_0 - \hat{\phi}_A(t_0, t_f)x_f)\nabla s, \\ &= \hat{\phi}_A(t_f, t_0)x_0 \\ &\quad - \hat{\phi}_A(t_f, t_0)\int_{t_0}^{t_f} \hat{\phi}_A(t_0, \rho(s))B(s)B^T(s)\hat{\phi}_A(t_0, \rho(s))\nabla s\; \hat{\mathcal{G}}_C^{-1}(t_0, t_f)(x_0 - \hat{\phi}_A(t_0, t_f)x_f) \\ &= \hat{\phi}_A(t_f, t_0)x_0 - (\hat{\phi}_A(t_f, t_0)x_0 - x_f) \\ &= x_f, \end{aligned}$$

so that the state equation is controllable on $[t_0, t_f]$.

Conversely, suppose that the state equation is controllable, but for the sake of a contradiction, assume the matrix $\hat{\mathcal{G}}_C(t_0, t_f)$ is not invertible. If $\hat{\mathcal{G}}_C(t_0, t_f)$ is not invertible, then there exists a vector $x_a \neq 0$ such that

$$0 = x_a^T\hat{\mathcal{G}}_C(t_0, t_f)x_a = \int_{t_0}^{t_f} x_a^T\hat{\phi}_A(t_0, \rho(s))B(s)B^T(s)\hat{\phi}_A^T(t_0, \rho(s))x_a\nabla s. \tag{7.2}$$

But, the function in this expression is the nonnegative continuous function $||x_a^T\hat{\phi}_A(t_0, \rho(s))B(s)||^2$, and so it follows that

$$x_a^T\hat{\phi}_A(t_0, \rho(s))B(s) = 0, \qquad t \in (t_0, t_f]. \tag{7.3}$$

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However, the state equation is controllable on $[t_0, t_f]_{\mathbb{T}}$, and so choosing $x_0 = x_a + \hat{\phi}_A(t_0, t_f)x_f$, there exists an input signal $u_a(t)$ such that

$$x_f = \hat{\phi}_A(t_f, t_0)x_0 + \int_{t_0}^{t_f} \hat{\phi}_A(t_f, \rho(s))B(s)u_a(s)\nabla s,$$

which is equivalent to the equation

$$x_a = -\int_{t_0}^{t_f} \hat{\phi}_A(t_0, \rho(s))B(s)u_a(s)\nabla s.$$

Multiplying through by x_a^T and using (7.2) and (7.3) yields $x_a^T x_a = 0$, a contradiction. Thus, the matrix $\hat{\mathcal{G}}_C(t_0, t_f)$ is invertible. □

Before producing our feedback theorem, we need a couple of lemmas.

Lemma 7.1. *The ν-Hilger circle $\mathbb{H}_\nu$ is closed under the operation $\oplus_\nu$.*

Proof. Let $\alpha \in \mathbb{C}$ be such that $|\alpha| > 1$. Then $a = \frac{1-\alpha}{\nu} \in \mathbb{H}_\nu$ since $\left|\frac{1-\alpha}{\nu} - \frac{1}{\nu}\right| = \left|-\frac{\alpha}{\nu}\right| > \frac{1}{\nu}$. Similarly, let $\beta \in \mathbb{C}$ be such that $|\beta| > 1$, so that $b = \frac{1-\beta}{\nu} \in \mathbb{H}_\nu$. We set

$$c := a \oplus_\nu b = a + b - \nu ab.$$

Now, $c \in \mathbb{H}_\nu$ if there exists a $\gamma \in \mathbb{C}$ such that $|\gamma| > 1$ with $c = \frac{1-\gamma}{\nu}$. We claim that the choice $\gamma = \alpha\beta$ will suffice, from which the claim follows immediately. Indeed, with this choice of γ, we have that

$$\frac{1-\gamma}{\nu} = \frac{1-\alpha}{\nu} + \frac{1-\beta}{\nu} - \nu\frac{1-\alpha}{\nu}\frac{1-\beta}{\nu},$$

and since $|\gamma| = |\alpha| \cdot |\beta| > 1$, the claim follows. □

Lemma 7.2 (Stability Under Change of State Variables)**.** *The ν-regressive nabla linear state equation*

$$\begin{aligned} x^\nabla(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \\ y(t) &= C(t)x(t), \end{aligned}$$

is ν-uniformly exponentially stable with rate $(\lambda + \alpha)/(1 - \nu_*\alpha)$, where $\lambda, \alpha > 0$ and $\alpha \in \mathcal{R}_\nu^+$, if the linear state equation

$$z^\nabla(t) = [(1 - \nu(t)\alpha)A(t) + \alpha I]z(t), \quad z(t_0) = x_0$$

is ν-uniformly exponentially stable with rate λ.

Proof. By direct calculation, $x(t)$ satisfies

$$x^\nabla(t) = A(t)x(t), \quad x(t_0) = x_0,$$

if and only if $z(t) = \hat{e}_\alpha(t, t_0)x(t)$ satisfies

$$z^\nabla(t) = [(1 - \nu(t)\alpha)A(t) + \alpha I]z(t), \quad z(t_0) = x_0. \tag{7.4}$$

Now assume there exists a $\gamma > 0$ such that for any x_0 and t_0, the solution of (7.4) satisfies

$$||z(t)|| \le \gamma\hat{e}_{-\lambda}(t, t_0)||x_0||, \quad t \ge t_0.$$

Then substituting for $z(t)$ yields

$$||\hat{e}_\alpha(t, t_0)x(t)|| = \hat{e}_\alpha(t, t_0)||x(t)|| \le \gamma\hat{e}_{-\lambda}(t, t_0)||x_0||,$$

so that

$$||x(t)|| \le \gamma\hat{e}_{-\lambda\ominus_\nu\alpha}(t, t_0)||x_0|| \le \gamma\hat{e}_{-(\lambda+\alpha)/(1-\nu_*\alpha)}(t, t_0)||x_0||,$$

where we note that $-(\lambda + \alpha)/(1 - \nu_*\alpha) \in \mathcal{R}_\nu^+$. □

We defined the controllability Gramian $\hat{\mathcal{G}}_C(t, \mathcal{C}(t))$ earlier as

$$\hat{\mathcal{G}}_C(t_0, t_f) = \int_{t_0}^{t_f} \hat{\phi}_A(t_0, \rho(s))B(s)B^T(s)\hat{\phi}_A^T(t_0, \rho(s))\nabla s. \tag{7.5}$$

To obtain the feedback result, we need to use the following shifted version of this matrix. For $\alpha > 0 \in \mathcal{R}_\nu^+$, define the matrix

$$\hat{\mathcal{G}}_{C_\alpha}(t_0, t_f) = \int_{t_0}^{t_f} (\hat{e}_\alpha(t_0, s))^4 \hat{\phi}_A(t_0, \rho(s)) B(s) B^T(s) \hat{\phi}_A^T(t_0, \rho(s)) \nabla s. \tag{7.6}$$

Theorem 7.2 (Gramian Exponential Stability Criterion)**.** *Let $\mathbb{T}$ be a time scale with bounded graininess. For the ν-regressive nabla linear state equation*

$$\begin{aligned} x^\nabla(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \\ y(t) &= C(t)x(t), \end{aligned}$$

suppose there exist positive constants ϵ_1, ϵ_2 and a strictly increasing function $\mathcal{C} : \mathbb{T} \to \mathbb{T}$ such that $0 < \mathcal{C}(t) - t \leq M < \infty$ with

$$\epsilon_1 I \leq \hat{\mathcal{G}}_C(t, \mathcal{C}(t)) \leq \epsilon_2 I, \tag{7.7}$$

for all t. Then given a positively regressive constant $\alpha > 0$, the state feedback gain

$$K(t) = -B^T(t)(I - \nu(t)A^T(t))^{-1} \hat{\mathcal{G}}_{C_\alpha}^{-1}(t, \mathcal{C}(t)), \tag{7.8}$$

is such that the resulting closed-loop state equation is uniformly exponentially stable with rate α.

Proof. We first note that for $N = \inf\limits_{t \in \mathbb{T}} \dfrac{\log(1 - \nu(t)\alpha)}{\nu(t)}$, we have $-\infty < N < 0$ since $\mathbb{T}$ has bounded graininess. Thus,

$$\begin{aligned} \hat{e}_\alpha(t, \mathcal{C}(t)) &= \exp\left(\int_t^{\mathcal{C}(t)} \frac{\log(1 - \nu(\tau)\alpha)}{\nu(\tau)} \nabla \tau \right) \\ &\geq \exp\left(\int_t^{\mathcal{C}(t)} N \nabla \tau \right) \\ &= e^{N(\mathcal{C}(t) - t)} \\ &\geq e^{MN}, \quad (\text{since } N < 0). \end{aligned}$$

Comparing the quadratic forms $x^T \hat{\mathcal{G}}_{C_\alpha}(t, \mathcal{C}(t)) x$ and $x^T \hat{\mathcal{G}}_C(t, \mathcal{C}(t)) x$ using their respective definitions (7.5) and (7.6) gives

$$e^{4MN} \hat{\mathcal{G}}_C(t, \mathcal{C}(t)) \leq \hat{\mathcal{G}}_{C_\alpha}(t, \mathcal{C}(t)) \leq \hat{\mathcal{G}}_C(t, \mathcal{C}(t)),$$

for all t. Thus, (7.7) gives

$$\epsilon_1 e^{4MN} I \leq \hat{\mathcal{G}}_\alpha(t, \mathcal{C}(t)) \leq \epsilon_2 I \tag{7.9}$$

for all t, and so the existence of $\hat{\mathcal{G}}_{C_\alpha}^{-1}(t, \mathcal{C}(t))$ is immediate. Now, we show that the linear state equation

$$z^\nabla(t) = [(1 - \nu(t)\alpha)\hat{A}(t) + \alpha I] z(t), \tag{7.10}$$

where $\hat{A}(t) - B(t)B^T(t)(I - \nu(t)A^T(t))^{-1} \hat{\mathcal{G}}_{C_\alpha}^{-1}(t, \mathcal{C}(t))$, is ν-uniformly exponentially by Theorem 6.3 with the choice

$$Q(t) = \hat{\mathcal{G}}_{C_\alpha}^{-1}(t, \mathcal{C}(t)). \tag{7.11}$$

Lemma 7.2 then yields the result. To apply the theorem, we first note that $Q(t)$ is symmetric and continuously nabla differentiable. Thus, (7.9) gives

$$\frac{1}{\epsilon_2} I \leq Q(t) \leq \frac{e^{-4MN}}{\epsilon_1} I, \tag{7.12}$$

for all t. Hence, it only remains to show that there exists $\gamma > 0$ such that

$$\frac{Q(t) - (I - \nu(t)\hat{A}^T(t)) Q(\rho(t)) (I - \nu(t)\hat{A}(t))}{\nu(t)} \leq -\gamma I.$$

We begin with the second term, writing

$$\begin{aligned}&\left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}(t)+\alpha I]^T\right]Q(\rho(t))\left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}(t)+\alpha I]\right]\\ &=(1-\nu(t)\alpha)^2\left[[I-\nu(t)A^T(t)]+\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))[I-\nu(t)A(t)]^{-1}\nu(t)B(t)B^T(t)\right]\\ &\cdot\hat{\mathcal{G}}_{C_\alpha}^{-1}(\rho(t),C(\rho(t)))\left[[I-\nu(t)A(t)]+\nu(t)B(t)B^T(t)[I-\nu(t)A^T(t)]^{-1}\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))\right].\end{aligned}$$

We pause to establish an important identity. Notice that

$$\begin{aligned}&[I-\nu(t)A(t)]\hat{\mathcal{G}}_{C_\alpha}(t,\mathcal{C}(t))[I-\nu(t)A^T(t)]\\ &=\frac{1}{(1-\nu(t)\alpha)^4}\hat{\mathcal{G}}_{C_\alpha}(\rho(t),\mathcal{C}(t))-\nu(t)B(t)B^T(t).\end{aligned}\tag{7.13}$$

This leads to

$$\begin{aligned}&I+\nu(t)[I-\nu(t)A(t)]^{-1}B(t)B^T(t)[I-\nu(t)A^T(t)]^{-1}\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))\\ &=\frac{1}{(1-\nu(t)\alpha)^4}[I-\nu(t)A(t)]^{-1}\hat{\mathcal{G}}_{C_\alpha}(\rho(t),\mathcal{C}(t))[I-\nu(t)A^T(t)]^{-1}\\ &\cdot\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t)),\end{aligned}\tag{7.14}$$

which in turn yields

$$\begin{aligned}&I+\nu(t)\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))[I-\nu(t)A(t)]^{-1}B(t)B^T(t)[I-\nu(t)A^T(t)]^{-1}\\ &=\frac{1}{(1-\nu(t)\alpha)^4}\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))[I-\nu(t)A(t)]^{-1}\hat{\mathcal{G}}_{C_\alpha}(\rho(t),\mathcal{C}(t))\\ &\cdot[I-\nu(t)A^T(t)]^{-1}.\end{aligned}\tag{7.15}$$

The second term can now be rewritten as

$$\begin{aligned}&&(1-\nu(t)\alpha)^2\left[[I-\nu(t)A^T(t)]+\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))[I-\nu(t)A(t)]^{-1}\nu(t)B(t)B^T(t)\right]\\ &\cdot&\hat{\mathcal{G}}_{C_\alpha}^{-1}(\rho(t),C(\rho(t)))\left[[I-\nu(t)A(t)]+\nu(t)B(t)B^T(t)[I-\nu(t)A^T(t)]^{-1}\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))\right]\\ &=&(1-\nu(t)\alpha)^2\left[I+\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))[I-\nu(t)A(t)]^{-1}\nu(t)B(t)B^T(t)[I-\nu(t)A^T(t)]^{-1}\right]\\ &\cdot&[I-\nu(t)A^T(t)]\hat{\mathcal{G}}_{C_\alpha}^{-1}(\rho(t),\mathcal{C}(\rho(t)))[I-\nu(t)A(t)]\\ &\cdot&\left[I+[I-\nu(t)A(t)]^{-1}\nu(t)B(t)B^T(t)[I-\nu(t)A^T(t)]^{-1}\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))\right].\end{aligned}$$

Using (7.14) and (7.15), we can now write

$$\begin{aligned}&&\left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}(t)+\alpha I]^T\right]Q(\rho(t))\left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}(t)+\alpha I]\right]\\ &=&(1-\nu(t)\alpha)^{-6}\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))[I-\nu(t)A(t)]^{-1}\hat{\mathcal{G}}_{C_\alpha}(\rho(t),\mathcal{C}(t))\hat{\mathcal{G}}_{C_\alpha}^{-1}(\rho(t),\mathcal{C}(\rho(t)))\\ &\cdot&\hat{\mathcal{G}}_{C_\alpha}(\rho(t),\mathcal{C}(t))[I-\nu(t)A^T(t)]^{-1}\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t)).\end{aligned}\tag{7.16}$$

On the other hand, from the definition of $\hat{\mathcal{G}}_{C_\alpha}(t,\mathcal{C}(t))$, we have

$$\hat{\mathcal{G}}_{C_\alpha}(\rho(t),C(\rho(t)))\le\hat{\mathcal{G}}_{C_\alpha}(\rho(t),C(t)),$$

which in turn implies

$$\hat{\mathcal{G}}_{C_\alpha}^{-1}(\rho(t),C(\rho(t)))\ge\hat{\mathcal{G}}_{C_\alpha}^{-1}(\rho(t),\mathcal{C}(t)).$$

Combining this with (7.16) gives

$$\begin{aligned}&&\left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}^T(t)+\alpha I]\right]Q(\rho(t))\left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}(t)+\alpha I]\right]\\ &\ge&(1-\nu(t)\alpha)^{-6}\hat{\mathcal{G}}_{C_\alpha}^{-1}(t,\mathcal{C}(t))\left[[I-\nu(t)A]^{-1}\hat{\mathcal{G}}_{C_\alpha}(\rho(t),\mathcal{C}(t))[I-\nu(t)A^T(t)]\right]\\ &\cdot&\hat{\mathcal{G}}_{C_\alpha}(t,\mathcal{C}(t)).\end{aligned}$$

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Applying (7.13) again yields

$$\begin{aligned}
& \left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}^T(t)+\alpha I]\right] Q(\rho(t)) \left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}(t)+\alpha I]\right] \\
\geq \quad & (1-\nu(t)\alpha)^{-6}\hat{\mathcal{G}}^{-1}_{C_\alpha}(t,\mathcal{C}(t)) \\
\cdot \quad & \left[(1-\nu(t)\alpha)^4\hat{\mathcal{G}}_{C_\alpha}(t,\mathcal{C}(t))+\nu(t)(1-\nu(t)a)^4[I-\nu(t)A(t)]^{-1}B(t)B^T(t)[I-\nu(t)A^T(t)]^{-1}\right] \\
\cdot \quad & \hat{\mathcal{G}}^{-1}_{C_\alpha}(t,\mathcal{C}(t)) \\
\geq \quad & (1-\nu(t)\alpha)^2\hat{\mathcal{G}}^{-1}_{C_\alpha}(t,\mathcal{C}(t)).
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{Q(t)-\left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}^T(t)+\alpha I]\right] Q(\rho(t)) \left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}(t)+\alpha I]\right]}{\nu(t)} \\
\leq \quad & -\frac{1-(1-\nu(t)\alpha)^2}{\nu(t)(1-\nu(t)\alpha)^2}\hat{\mathcal{G}}^{-1}_{C_\alpha}(t,\mathcal{C}(t)) \\
\leq \quad & -\frac{1-(1-\nu(t)\alpha)^2}{\nu(t)(1-\nu(t)\alpha)^2\epsilon_2}I.
\end{aligned}$$

Now, the quantity $(1-(1-\nu(t)\alpha)^2)/(\nu(t)(1-\nu(t)\alpha)^2\epsilon_2)$ is certainly not constant, but it can be bounded by a quantity that is (here $\nu_*=\nu_{\min}$):

$$\frac{1-(1-\nu(t)\alpha)^2}{\nu(t)(1-\nu(t)\alpha)^2\epsilon_2}=\frac{2\alpha-\nu(t)\alpha^2}{(1-\nu(t)\alpha)^2\epsilon_2}\geq\frac{2\alpha-\nu_*\alpha^2}{(1-\nu_*\alpha)^2\epsilon_2}.$$

Thus, if we set $\gamma=\frac{2\alpha-\nu_*\alpha^2}{(1-\nu_*\alpha)^2\epsilon_2}$, then

$$\frac{Q(t)-\left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}^T(t)+\alpha I]\right] Q(\rho(t)) \left[I-\nu(t)[(1-\nu(t)\alpha)\hat{A}(t)+\alpha I]\right]}{\nu(t)}\leq-\gamma I.$$

□

At this point, it is worth discussing possible choices for the function $\mathcal{C}(t)$ which we term the **compactification operator**. If $\mathbb{T}$ is purely discrete (i.e. has no points with $\nu(t)=0$), then one possible choice for $\mathcal{C}(t)$ is $\mathcal{C}(t)=\sigma^k(t)$ for some $k\in\mathbb{N}$. For $\mathbb{T}=\mathbb{R}$, it is well known that the choice $\mathcal{C}(t)=t+\delta$, for some $\delta>0$ will suffice. If $\mathbb{T}=\mathbb{P}_{a,b}$ (a disjoint union of closed intervals of length a and gaps between intervals of length b), then the choice $\mathcal{C}(t)=t+a+b$ is a possibility. These examples show that the choice of the compactification operator can vary widely with the time scale involved, and so this is why we cast the theorem in terms of a general operator.

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