Time Scale Discrete Fourier Transforms

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Abstract—The discrete and continuous Fourier transforms are applicable to discrete and continuous time signals respectively. Time scales allows generalization to to any closed set of points on the real line. Discrete and continuous time are special cases. Using the Hilger exponential from time scale calculus, the discrete Fourier transform (DFT) is extended to signals on a set of points with arbitrary spacing. A time scale \( \mathbb{D}_N \) consisting of \( N \) points in time is shown to impose a time scale (more appropriately dubbed a frequency scale), \( \mathbb{U}_N \), in the Fourier domain. The time scale DFT’s (TS-DFT’s) are shown to share familiar properties of the DFT, including the derivative theorem and the power theorem. Shifting on a time scale is accomplished through a boxplus and boxminus operators. The shifting allows formulation of time scale convolution and correlation which, as is the case with the DFT, correspond to multiplication in the frequency domain.

I. INTRODUCTION

A time scale is any collection of closed points on the real line. Continuous time, \( \mathbb{R} \), and discrete time \( \mathbb{Z} \), are special cases. The calculus of time scales was introduced by Hilger [11]. Time scales have found utility in describing the behavior of dynamic systems [1], [13] and have been applied to control theory [3], [4], [5], [7], [10].

On \( \mathbb{R} \) and \( \mathbb{Z} \), respectively, the continuous time and discrete time Fourier transforms are well studied [16]. Properties of the Laplace and Fourier transforms on time scales have been extended to time scales with unbounded domains [1], [6], [8], [9], [12], [14], [16].

The conventional discrete Fourier transform (DFT) is defined over a finite number of uniformly spaced points. This paper extends the DFT to a finite number of discrete time points that are not uniformly spaced.\(^1\) The time scale of a finite number of \( N \) discrete points, \( \mathbb{D}_N \), is shown to uniquely map into a frequency scale (a time scale in the frequency domain), \( \mathbb{U}_N \), in the Fourier domain. Familiar Fourier transform theorems, including the shift, convolution and derivative theorems, are shown to generalize to the time scale DFT (TS-DFT).

II. TIME SCALES

Our introduction to time scales is limited to that needed to establish notation. A more detailed explanation are in our previous papers [4], [5], [6], [8], [9], [10], [13], [14], [16] and a complete rigorous treatment is in the text by Bohner and Peterson [1].

1) A time scale, \( \mathbb{T} \), is any collection of closed intervals on the real line. Generally, the time scale can contain both discrete time points and continuous time intervals. Since our development of TS-DFT is only on time scales containing discrete points, we henceforth restrict attention to time scales containing discrete points, denoted \( \mathbb{D} \). Discrete time, \( \mathbb{Z} \), is a special case.

2) The graininess, \( \mu(t) \), is the distance between adjacent points in a time scale at time \( t \in \mathbb{T} \) and is defined in general by

\[
\mu(t) = \inf_{\tau > t, \tau \in \mathbb{T}} (\tau - t).
\]

For \( \mathbb{D} \),

\[
\mu(t_n) = t_{n+1} - t_n.
\]

3) The Hilger derivative of a function \( x(t) \) at \( t \in \mathbb{T} \) is

\[
x^{\Delta}(t) := \frac{x(t + \mu(t)) - x(t)}{\mu(t)}.
\]

When \( \mu(t) = dt \) \( (= 0) \), the Hilger derivative is interpreted in the limiting sense and

\[
x^{\Delta}(t) = \frac{d}{dt} x(t).
\]

For \( \mathbb{D} \), we have

\[
x^{\Delta}(t_n) = \frac{x(t_{n+1}) - x(t_n)}{\mu(t_n)}.
\]

4) If \( y(t) = x^{\Delta}(t) \), then the definite time scale integral is

\[
\int_a^b y(t) \Delta t = z(b) - z(a).
\]

For \( \mathbb{D} \), we have [1]

\[
\int_{t_p}^{t_q} y(t) \Delta t = \sum_{n=p}^{q-1} y(t_n) \mu(t_n).
\]

5) When \( x(0) = 1 \), the solution to the Hilger differential equation,

\[
x^{\Delta}(t) = x(t),
\]

\(^2\)We use \( \mathbb{D} \) to denote a time scale with an arbitrary, possibly infinite, set of discrete isolated points. The notation \( \mathbb{D}_N \) indicates the time scale has \( N \) points.

\(^1\)Our development is distinct from the time scale Fourier transform proposed by Hilger [12], [14], [16]. Our treatment more closely resembles Laplace transform generalizations where two signals on a time scale \( \mathbb{T} \), when convolved, result in a signal on the same time scale, \( \mathbb{T} \) [1], [6], [8], [9].
is \( x(t) = e_z(t) \) where the generalized exponential is
\[
e_z(t) := \exp \left( \int_{\tau=0}^{t} \frac{\ln (1 + z\mu(\tau))}{\mu(\tau)} \Delta \tau \right).
\]
For \( \mathbb{D} \) and \( n > 0 \),
\[
e_z(t_n) = \prod_{m=0}^{n-1} (1 + z\mu(t_m)). \tag{1}
\]
Since \( \mu(t_m) \) is real,
\[
e_z(t_n) = e_z^r(t_n) \tag{2}
\]
The properties of the generalized exponential parallel those of \( z^n \) for the z-transform and \( e^{iw t} \) for the Fourier transform are responsible for the utility of the TS-DFT.

### III. Time Scale Exponential Basis Sets

Consider a time scale, \( \mathbb{D}_N \), of \( N + 1 \) real temporal points, \( \{t_n|0 \leq n \leq N\} \) with \( t_0 \leq t_n < t_{n+1} \leq t_N \). The point \( t_N \) is required to determine the disjointness of the point \( t_{N-1} \). Let \( x(t_n) \) and \( h(t_n) \) be images on \( \mathbb{D}_N \). Define the inner product
\[
\langle x(t_n) | h(t_n) \rangle = \sum_{n=0}^{N-1} x(t_n)h^*(t_n)w(t_n) \tag{3}
\]
where \( w(t_n) > 0 \) is a weighting function and the asterisk denotes complex conjugation. Generally, \( w(t_n) \) is arbitrary but, to make the integration constant with time scale integration, we will henceforth use the graining as the weight, i.e. \( w(t_n) = \mu(t_n) \). The corresponding norm is
\[
||x(t)|| = \sqrt{\langle x(t_n) | x(t_n) \rangle}.
\]
Two time scale exponentials are orthogonal if
\[
\langle e_z(t_n) | e_z^r(t_n) \rangle = 0 \text{ for } z \neq \zeta. \tag{4}
\]
If we set \( \zeta = 0 \), then applying (1) and (3) to (4) gives
\[
\langle e_z(t_n) | e_z^r(t_n) \rangle|_{\zeta=0} = \langle e_z(t_n) | 1 \rangle
\]
\[
= \sum_{n=0}^{N-1} e_z(t_n)\mu(t_n)
\]
\[
= \mu(t_0) + \sum_{n=1}^{N-1} \mu(t_n) \prod_{m=0}^{n-1} (1 + z\mu(t_m))
\]
\[
= 0 \text{ for } z \neq \zeta = 0. \tag{5}
\]
The solution of the \( N - 1 \)st order polynomial can be used to generate orthogonal time scale exponentials. Motivated by (5), we dub the roots of the polynomial
\[
\mu(t_0) + \sum_{n=1}^{N-1} \mu(t_n) \prod_{m=0}^{n-1} (1 + z\mu(t_m)) = 0 \tag{6}
\]
the frequency roots of the time scales, \( \mathbb{D}_N \).

#### A. Example Frequency Roots

Here are some examples of frequency roots from the polynomial in (6).

1) **On a Time Scale of Uniformly Spaced Points**

The time scale of uniformly spaced points is the time scale conventionally associated with the DFT. Then \( \mu(t_n) = 1 \) and (5) becomes
\[
\langle e_z(t_n) | 1 \rangle = 1 + \sum_{n=1}^{N-1} \prod_{m=0}^{n-1} (1 + z)
\]
\[
= \sum_{n=0}^{N-1} (1 + z)^n = 0.
\]
This is a geometric series with solution
\[
\langle e_z(t_n) | 1 \rangle = (1 + z)^N - 1 = 0. \tag{7}
\]
Note that the zeroth order term in the numerator is zero, so since \( z \neq \zeta = 0 \), (7) is an \((N - 1)\)st polynomial with \( N - 1 \) frequency roots. For \( z \neq \zeta = 0 \) (required by (4)) and \( z \neq -1 \) (the regressive condition for \( \mathbb{Z} \)), this equation is satisfied when \( (1 + z)^N = \exp(-j2\pi k) \) where \( k \) is an arbitrary integer.

Thus the \( N - 1 \) polynomial frequency roots are
\[
z_k = -1 + e^{-j2\pi k/N}; \quad 0 < k < N. \tag{8}
\]
As shown in Figure 1, these are points equally spaced on a unit circle centered at \( z = -1 \).

Other time scales do not lend themselves to the ease of analysis afforded by the time scale of uniformly spaced points.

2) **On a log time scale**

we have \( \mathbb{D}_N = \{t_n = \log_2(n)\} \). Example frequency roots of this time scale are shown in Figure 1.

3) **The Harmonic Time Scale**

is defined as
\[
t_n = \begin{cases} 
0 & ; n = 0 \\
\sum_{k=1}^{n} \frac{1}{k} & ; n > 0.
\end{cases}
\]

The frequency roots are shown in Figure 1 for \( N = 16 \).

4) **The Geometric Time Scale**

for a parameter \( q > 0 \), is defined as
\[
t_n = \begin{cases} 
0 & ; n = 0 \\
q^n & ; n > 0.
\end{cases}
\]

The frequency roots are shown in Figure 2 for \( N = 16 \).

For \( q < 1 \), the values of the time scale are the same as in (9) except they are arranged in ascending order.

5) **The Poisson Time Scale**

chooses points in a Poisson process [16] with parameter \( \lambda \). The origin, \( t_0 = 0 \), is then included. Example frequency roots are shown in Figure 3.

#### B. Basis Examples

Here are some examples of time scale exponential basis sets for some example time scales. For the exponential basis plots in Figure 4, \( N = 8 \). Points are linearly connected. The location of points on the time scale are marked along the time axis with dots.

1) **On a Time Scale with Uniformly Spaced Points**

the TS-DFT becomes the conventional DFT [16]. For the uniformly

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\( ^3 \)This differs from the quantum time scale [2] which includes the origin and points \( q^n \) for \( n \in \mathbb{Z} \).
Fig. 1. Frequency roots of some time scales. In each plot, the horizontal and vertical scales are the same. LEFT: The frequency roots of the time scales with equally spaced points lie on a shifted circle in the $z$ plane, dubbed the Hilger circle [1]. MIDDLE: The frequency roots of the log time scale, plotted on the complex $z$ plane, discussed in Section III-A1 for $N = 16$. RIGHT: The frequency roots of the harmonic scale, plotted on the complex $z$ plane, as discussed in Section III-A1 for $N = 16$.

Fig. 3. The frequency roots of 15 realizations of a Poisson time scale with parameter $\lambda = 1$ point per interval. The shape of the root locations varies considerably. Note that scales can differ from plot to plot.

spaced points described in (III-A1), the orthonormalized exponential basis functions are

$$e_{z_k}(t_n) = e^{j2\pi nk/N}; \text{ for } 0 \leq n, k < N. \tag{9}$$

Proof: Since $\mu(t_n) = 1$, we have from (1)

$$e_{z_k}(t_n) = \prod_{m=0}^{n-1} (1 + z) = (1 + z)^n.$$

Substituting (8) gives (9). These are the familiar normalized basis functions for the discrete Fourier transform (DFT) and are shown in Figure 4.

1) Other Basis Sets: The orthonormalized basis functions for the log time scale, the harmonic time scale, and the geometric time scale for $q = 1.2$ are shown in Figure 4.

C. Orthogonal Expansions and Inversion

When the orthogonal basis $\{e_{z_k}(t_n)|0 \leq n, k < N\}$ is complete, we can expand any function, $x(t_n)$, on the time scale as

$$x(t_n) = \sum_{\ell=0}^{N-1} c_{\ell} e_{z_k}(t_n)$$

where $c_{n}$ are the series expansion coefficients. Let

$$X(z_k) := \sum_{n=0}^{N-1} x(t_n)e_{z_k}^*(t_n)\mu(t_n)$$

$$= \sum_{n=0}^{N-1} \left[ \sum_{\ell=0}^{N-1} c_{\ell} e_{z_k}(t_n) \right] e_{z_k}^*(t_n)\mu(t_n)$$

$$= \sum_{\ell=0}^{N-1} c_{\ell} \left[ \sum_{n=0}^{N-1} e_{z_k}(t_n)e_{z_k}^*(t_n)\mu(t_n) \right]$$

$$= c_k \| e_{z_k} \|^2.$$

Thus

$$c_k = \frac{X(z_k)}{\| e_{z_k} \|^2}$$

where

$$\| e_{z_k} \|^2 := \sum_{n=0}^{N-1} |e_{z_k}(t_n)|^2\mu(t_n).$$
Fig. 2. TOP: The frequency roots of the geometric time scale in (9), plotted on the complex $z$ plane, as discussed in Section III-A1 for $N = 16$. The values of $q$ are $1.20 \oplus, 1.25 \odot, 1.30 \ominus$, and $1.35 \oslash$. BOTTOM: The frequency roots of the geometric time scale, plotted on the complex $z$ plane, as discussed in Section III-A1 for $N = 16$. The values of $q$ are $0.85 \odot, 0.875 \oplus, 0.90 \ominus$, and $0.99 \oslash$.

We find the following notation useful.\(^4\)

$$
\partial t_n := \mu(t_n)
$$

and

$$
\partial z_k := \frac{1}{\|e_{zk}\|^2}.
$$

(10)

Note that if $t_n$ has units of time, then $\partial t_n$ has units of time and $\partial z_k$ has units of reciprocal time.

From this analysis, we define the time scale DFT (TS-DFT) and its inverse.

\[\text{TS-DFT}\]

$$
x(t_n) \leftrightarrow X(z_k) = \sum_{n=0}^{N-1} x(t_n) e_{zk}^* (t_n) \partial t_n.
$$

(11)

\[\text{Inverse TS-DFT}\]

$$
x(t_n) = \sum_{k=0}^{N-1} X(z_k) e_{zk} (t_n) \partial z_k \leftrightarrow X(z_k).
$$

(12)

\(^4\)An alternate possibly more representative notation might be $\mu_0(t_n)$ in lieu of $\partial t_n$ and $\mu_0(z_k)$ instead of $\partial z_k$. We have opted for the shorter more compact notation.

Thus $x(t_n)$ is a finite duration signal on a time scale $\mathbb{D}_N$ with graininess $\partial t_n = \mu(t_n)$. This imposes a frequency scale, $\mathbb{U}_N$, with values $X(z_k)$ and graininess $\partial z_k$ given by (10). Thus

$$
\mu_k = \begin{cases} 
0; & k = 0 \\
\sum_{\ell=1}^{k} \partial z_\ell = z_{k-1} + \partial z_k; & 1 \leq k \leq N.
\end{cases}
$$

(13)

define the point locations on the time scale $\mathbb{U}_N$. The image $X(z_k)$ is assigned to the point\(^5\) $u_k$.

\[\text{Conjugate Symmetry.}\] When $x(t_n)$ is real, $X^*(z_k) = X(z_k^*)$.

\[\text{Proof.}\] The proof results immediately upon applying (2) to the TS-DFT definition in (11).

D. TS-DFT Transform Theorems

Here are some theorems that parallel the conventional DFT [16].

\[\text{Area Theorem.}\] Since $t_0 = 0$,

$$
X(z_0) = \sum_{n=0}^{N-1} x(t_n) \partial t_n
$$

(14)

\[\text{Conjugate Symmetry.}\] When $x(t_n)$ is real, $X^*(z_k) = X(z_k^*)$.

\[\text{Proof.}\] The proof results immediately upon applying (2) to the TS-DFT definition in (11).

\(^5\)An alternate notation might be $X(u_k)$ in lieu of $X(z_k)$. We choose to continue with the notation $X(z_k)$.
Fig. 4. The real (top) and imaginary (bottom) components of the orthonormalized basis functions for the linear time scale using $N = 8$. The LINEAR plots are the familiar sins and cosines of the DFT kernel. The GEOMETRIC time scale is for $q = 1.2$.

Proof: Since $z_0 = 0$ and $e_z(0) = 1$, this follows immediately from the TS-DFT definition in (11). Likewise,

$$x(t_0) = \sum_{k=0}^{N-1} X(z_k) \partial z_k.$$  

Conjugate Symmetry. When $x(t_n)$ is real, $X^*(z_k) = X(z_k^*)$.

Proof: The proof results immediately upon applying (2) to the TS-DFT definition in (11).

Power Theorem.

$$\sum_{n=0}^{N-1} x(t_n) h^*(t_n) \partial t_n = \sum_{k=0}^{N-1} X(z_k) H^*(z_k) \partial z_k. \quad (14)$$

Proof: Follows from the definition of the TS-DFT in (11) and its inverse in (12).

$$\sum_{k=0}^{N-1} X(z_k) H^*(z_k) \partial z_k$$

$$= \sum_{k=0}^{N-1} X(z_k) \left[ \sum_{n=0}^{N-1} h(t_n) e_{z_k^*}(t_n) \partial t_n \right]^* \partial z_k$$

$$= \sum_{n=0}^{N-1} \left[ \sum_{k=0}^{N-1} X(z_k) e_{z_k}(t_n) \partial z_k \right] h^*(t_n) \partial t_n$$

$$= \sum_{n=0}^{N-1} x(t_n) h^*(t_n) \partial t_n$$

Parseval’s Theorem is a special case of the power theorem when $x = h$.

$$\sum_{n=0}^{N-1} ||x(t_n)||^2 \partial t_n = \sum_{k=0}^{N-1} ||X(z_k)||^2 \partial z_k.$$  

Derivative theorem.

$$x^\Delta(t_n) \leftrightarrow z_k X(z_k), \quad (15)$$
Fig. 5. A graphical illustration of the TS-DFT. On the upper left is a signal, 
\( x(t_n) \), on a time scale \( \mathbb{D}_N \) where, here, \( N = 8 \). The time scale \( \mathbb{D}_N \) dictates the frequency roots, \( z_k \), as illustrated in Figures 1, 2 and 3 and the exponential basis sets illustrated in Figure 4. The basis set applied to the signal \( x(t_n) \) gives the values of the TS-DFT, namely \( X(z_k) \), as illustrated in the bottom figure. The norms of the basis set components determines the \( \delta z_k \)'s in (10) which, in turn, determines the frequency scale, \( \mathbb{U}_N \), in (13). This is shown in the upper right.

**Proof:** From (12),

\[
x^\Delta(t_n) = \sum_{k=0}^{N-1} X(z_k) e_{z_k}^*(t_n) \delta z_k
\]

from which (15) follows.

**IV. Shifts on a Time Scale**

\( \Box \) The boxminus shift operator, \( \Box \), on an arbitrary function, \( h(t_n) \), on \( \mathbb{D}_N \), is defined by its TS-DFT.

\[
h(t_n \Box t_m) \leftrightarrow H(z_k)e_{z_k}^*(t_m).
\]

**The Hilger delta** [6] is defined as

\[
\delta(t_n) := \frac{\delta[n]}{\delta t_0}
\]

where \( \delta[n] \) is the Kronecker delta\(^6\) and we have used \( t_0 = 0 \).

**The TS-DFT of the Hilger delta** is

\[
\delta(t_n) \leftrightarrow 1.
\]

**Proof:** The proof follows directly from the TS-DFT in (11).

\( \Box \) The shifted Hilger delta and its TS-DFT is

\[
\delta(t_n \Box t_m) = \frac{\delta[n - m]}{\delta t_m} \leftrightarrow e_{z_k}^*(t_m).
\]

**Proof:** Follows from application of (16) to (17).

\(^6\)\( \delta[n] = 1 \) for \( n = 0 \) and is otherwise zero.

**TABLE I**

Some TS-DFT Theorems. All sums are from 0 to \( N - 1 \), i.e., over \( n \)
we have \( \sum = \sum_{n=0}^{N-1} \) and, over \( k \), \( \sum_{k=0}^{N-1} \). (A) Convolution is defined in (28) and (B) Correlation in (30).

<table>
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<th>Theorem</th>
<th>Formula</th>
</tr>
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<tr>
<td>TS-DFT</td>
<td>( x(t_n) ) on ( \mathbb{D}_N ) \leftrightarrow ( X(z_k) ) on ( \mathbb{U}_N )</td>
</tr>
<tr>
<td>Transform</td>
<td>( X(z_k) = \sum x(t_n)e_{z_k}^*(t_n) \delta t_n )</td>
</tr>
<tr>
<td>Inverse</td>
<td>( x(t_n) = \sum X(z_k)e_{z_k}(t_n) \delta z_k )</td>
</tr>
<tr>
<td>Area theorem</td>
<td>( \sum x(t_n) \delta t_n = X(z_0) )</td>
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<tr>
<td>Symmetry (x real)</td>
<td>( X^<em>(z_k) = X(z_k^</em>) )</td>
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<tr>
<td>Parseval’s theorem</td>
<td>( \sum</td>
</tr>
<tr>
<td>Box minus shift theorem</td>
<td>( h(t_n \Box t_m) \leftrightarrow H(z_k)e_{z_k}^*(t_m) )</td>
</tr>
<tr>
<td>Inverted ( \Box ) shift theorem</td>
<td>( x^<em>(t_m \Box t_n) \leftrightarrow X^</em>(z_k)e_{z_k}^*(t_m) )</td>
</tr>
<tr>
<td>Box minus theorem</td>
<td>( x^<em>(\Box t_n) \leftrightarrow X^</em>(z_k) )</td>
</tr>
<tr>
<td>Box plus shift theorem</td>
<td>( h(t_n \Box t_m) \leftrightarrow H(z_k)e_{z_k}(t_m) )</td>
</tr>
<tr>
<td>Convolution(^a)</td>
<td>( x(t_n) * h(t_n) \leftrightarrow X(z_k)H(z_k) )</td>
</tr>
<tr>
<td>Correlation(^b)</td>
<td>( x(t_n) \star h(t_n) \leftrightarrow X^*(z_k)H(z_k) )</td>
</tr>
<tr>
<td>Derivative</td>
<td>( x^\Delta(t_n) \leftrightarrow z_kX(z_k) )</td>
</tr>
<tr>
<td>Frequency response</td>
<td>( h(t_n) * e_{z_k}(t_n) = H(z_k)e_{z_k}(t_n) )</td>
</tr>
</tbody>
</table>

\(^a\) convolution

\(^b\) correlation

**Proof:** Substitute (20) into (11) and use the orthogonal property in (4).
\( e_{z_k}(t_n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \)

Hilger delta
\( \delta(t_n) := \delta[n]/\delta(t_0) \leftrightarrow 1 \)

shifted Hilger delta
\( \delta(t_n \boxplus t_m) \leftrightarrow e^*_{z_k}(t_m) \)

sifting property
\[ \sum_{m=0}^{N-1} x(t_m) \delta(t_n \boxplus t_m) \mu(t_m) = x(t_n) \]

convolution identity
\[ x(t_n) \ast \delta(t_n) = x(t_n) \]

one
1 \( \rightarrow \) \( \delta(z_k) = \delta[k]/\delta(t) \)

basis exponential
\( e_{z_k}(t_n) \leftrightarrow \delta(z_k \boxplus t_n) \)

conjugate symmetry
\( e^*_{z_k}(t_n) \leftrightarrow \delta(z_k \boxplus t_n) \)

box minus shift
\( e_{z_k}(t_n \boxminus t_m) = e_{z_k}(t_n) e^*_{z_k}(t_m) \)

box plus shift
\( e_{z_k}(t_n \boxplus t_m) = e_{z_k}(t_n) e_{z_k}(t_m) \)

<table>
<thead>
<tr>
<th>Box Plus Commutivity.</th>
<th>The commutative property of the box plus shift is</th>
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<tbody>
<tr>
<td>( x(t_n \boxplus t_m) = x(t_m \boxplus t_n) ).</td>
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### Proof:

\[ x(t_m \boxplus t_n) = \sum_{k=0}^{N-1} X(z_k) e_{z_k}(t_m \boxplus t_n) \partial z_k \]

Conjugating both sides gives (23).

A special case of (23) is
\[ x^*(\boxminus t_n) \leftrightarrow X^*(z_k) \]

### Box Plus Operation.

The box plus operation is defined as
\[ x(t_m \boxplus t_n) := x(t_m \boxplus (\boxminus t_n)) \]

### Box Plus Semigroup Property.

\[ e_{z_k}(t_n \boxplus t_m) = e_{z_k}(t_n) e_{z_k}(t_m) \].

### Proof:

\[ e_{z_k}(t_n \boxplus t_m) = e_{z_k}(t_n) e_{z_k}(t_m) \]

Interpreting \( e_{z_k}(t_m) = e_{z_k}(0 \boxplus t_m) \), it follows that
\[ e_{z_k}(t_m) = e_{z_k}(t_m) \).

### Box Plus Commutivity.

The commutative property of the box plus shift is
\[ x(t_n \boxplus t_m) = x(t_m \boxplus t_n) \].

### Proof:

\[ x(t_n \boxplus t_m) = \sum_{k=0}^{N-1} X(z_k) e_{z_k}(t_n \boxplus t_m) \partial z_k \]

From (24).

from which (26) immediately follows.

### Box Plus TS-DFT.

The TS-DFT of a box plus shift is
\[ x(t_n \boxplus t_m) \leftrightarrow X(z_k) e_{z_k}(t_m) \]

### Proof:

\[ x(t_n \boxplus t_m) = \sum_{k=0}^{N-1} X(z_k) e_{z_k}(t_n \boxplus t_m) \partial z_k \]

\[ = \sum_{k=0}^{N-1} [X(z_k) e_{z_k}(t_m)] e_{z_k}(t_n) \partial z_k \]

from which (27) follows.

### Box Plus Identity.

\[ x(\boxplus t_n) = x(t_n) \].

### Table II

**Properties of Exponentials and Hilger Deltas.**

<table>
<thead>
<tr>
<th>Exponential Shift</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Basis Exponential Shift.</strong> The box minus basis exponential shift can be written as</td>
</tr>
<tr>
<td>( e_{z_k}(t_n \boxminus t_m) = e_{z_k}(t_n) e^*_{z_k}(t_m) ).</td>
</tr>
</tbody>
</table>

### Proof:

Applying (16) to (20) gives
\[ e_{z_k}(t_n \boxminus t_m) \leftrightarrow e^*_{z_k}(t_m) \delta(z_k \boxplus z_t) = e^*_{z_k}(t_m) \delta(z_k \boxplus z_t) \]

But, from (20),
\[ e_{z_k}(t_n) e^*_{z_k}(t_m) \leftrightarrow e^*_{z_k}(t_m) \delta(z_k \boxplus z_t) \]

Since the transforms in both cases are the same, (21) follows.

Interpreting
\[ e_{z_k}(\boxplus t_m) = e_{z_k}(0 \boxplus t_m) \]

it follows from the basis exponential shift identity in (21) that
\[ e_{z_k}(\boxplus t_m) = e^*_{z_k}(t_m) \).

### The TS-DFT of an inverted box minus shift is
\[ x^*(t_m \boxplus t_n) \leftrightarrow X^*(z_k) e^*_{z_k}(t_m) \]

![Box Plus Operation.](image-url)
Proof: Using (25),
\[ x(⊞t_n) = \sum_{k=0}^{N-1} X(z_k)e_{z_k}(⊞t_n)∂z_k \]
\[ = \sum_{k=0}^{N-1} X(z_k)e_{z_k}(t_n)∂z_k = x(t_n). \]

V. TIME SCALE CONVOLUTION AND CORRELATION

\textbf{Discrete time scale convolution} between two functions is defined as
\[ x(t_n) * h(t_n) := \sum_{m=0}^{N-1} x(t_m)h(t_n □ t_m)∂t_m. \quad (28) \]

\textbf{The TS-DFT of a convolution} is the product of the transforms.
\[ x(t_n) * h(t_n) ↔ X(z_k)H(z_k). \quad (29) \]

Proof: Let \( y = x * h. \) Then
\[ Y(z_k) = \sum_{n=0}^{N-1} y(t_n)e_{z_k}^*(t_n)∂t_n \]
\[ = \sum_{n=0}^{N-1} \left[ \sum_{m=0}^{N-1} x(t_m)h(t_n □ t_m)∂t_m \right] e_{z_k}^*(t_n)∂t_n \]
\[ = \sum_{m=0}^{N-1} x(t_m) \left[ \sum_{n=0}^{N-1} h(t_n □ t_m)e_{z_k}^*(t_n)∂t_n \right]∂t_m \]
\[ = \sum_{m=0}^{N-1} \left[ \sum_{n=0}^{N-1} x(t_m)e_{z_k}^*(t_n)∂t_m \right] H(z_k) = \sum_{m=0}^{N-1} [X(z_k)e_{z_k}^*(t_n)] H(z_k). \]

\textbf{Discrete time scale convolution} is
- commutative
  \[ x * h = h * x, \]
- associative
  \[ g * (h * x) = (g * h) * x, \]
- and distributive over addition
  \[ x * (g + h) = x * g + x * h. \]

Proof: The proof follows immediately from the TS-DFT of a convolution in (29).

\textbf{Discrete time scale correlation} between two functions is defined as
\[ x(t_n) * h(t_n) := \sum_{m=0}^{N-1} x^*(t_m)h(t_n □ t_m)∂t_m. \quad (30) \]

\textbf{Transformation of Correlation.}
\[ x(t_n) * h(t_n) ↔ X^*(z_k)H(z_k). \quad (31) \]

\textbf{Correlation} obeys the following laws.

1) Reversal. If \( y = x * h, \) and \( λ = h * x, \) then
\[ λ(t_n) = y^*(⊞t_n). \]

2) Order
\[ g * (h * x) = h * (g * x) = (h * g) * x. \]

3) Distributive over addition
\[ x * (g + h) = x * g + x * h. \]

Proof: Let \( y(t_n) = x(t_n) * h(t_n). \) Then
\[ y(t_n) = \sum_{k=0}^{N-1} X(z_k)H(z_k)e_{z_k}(t_n)∂z_k, \]
I. convolution  \( x \ast h \leftrightarrow XH \)

II. correlation  \( x \ast h \leftrightarrow X^*H \)

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutative</td>
<td>( h \ast x = x \ast h )</td>
</tr>
<tr>
<td>Associative</td>
<td>( g \ast (h \ast x) = (g \ast h) \ast x )</td>
</tr>
<tr>
<td>Distributive</td>
<td>( x \ast (g + h) = x \ast g + x \ast h )</td>
</tr>
</tbody>
</table>

III. Reversal  \( y(t_n) = x \ast h \rightarrow h \ast x = y^*(\Xi t_n) \)

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>( g \ast (h \ast x) = h \ast (g \ast x) )</td>
</tr>
<tr>
<td></td>
<td>( (h \ast g) \ast x )</td>
</tr>
<tr>
<td>Distributive</td>
<td>( x \ast (g + h) = x \ast g + x \ast h )</td>
</tr>
</tbody>
</table>

IV. Correlation  \( x(t_n) \ast h(t_n) = x^*(\Xi t_n) \ast h(t_n) \)

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convolution</td>
<td>( x(t_n) \ast h(t_n) = x^*(\Xi t_n) \ast h(t_n) )</td>
</tr>
</tbody>
</table>

**TABLE III**


Thus

\[
y^*(t_n) = \sum_{k=0}^{N-1} X^*(z_k)H(z_k)e_{\Xi t_n}^* \partial z_k.
\]

2) Follows immediately from

\[
g \ast (h \ast x) \leftrightarrow G^*H^*X.
\]

3) Follows immediately from the definition of correlation in (30).

**The relationship between convolution and correlation.**

\[
x(t_n) \ast h(t_n) = x^*(\Xi t_n) \ast h(t_n)
\]

and

\[
x(t_n) \ast h(t_n) = x^*(\Xi t_n) \ast h(t_n).
\]

\[
g \ast (h \ast x) = h \ast (g \ast x) = (h \ast g) \ast x.
\]

**VI. FINAL REMARKS**

The TS-DFT establishes a generalization of the DFT to cases where points are not spaced uniformly. The generalization preserves properties of the conventional DFT, including derivative, shift, convolution and correlation relationships. The TS-DFT and its inverse are defined in (11) and (12). TS-DFT theorems are listed in Tables I, II and III.

Much work remains in the development of the foundations of the TS-DFT. The study of the mapping of time scales, \(\mathbb{D}_N\), to frequency scales, \(\mathbb{U}_N\), remains, for example, an open problem, as does filtering, sampling, and the mechanics of convolution and correlation [15].

**ACKNOWLEDGEMENT**

The authors are appreciative of John E. Miller’s review and comments on this manuscript. This work was supported by NSF award CMMI-726996.

**REFERENCES**